Team Play Topics

Round Two

The first section of the Round Two Mandelbrot Team Play is reproduced below. A list of topics and practice problems are also provided to aid in preparation. Note that these problems are not meant to serve as a precise indicator of the problems that will appear on the contest. However, students who understand how to solve them should be able to make significantly more progress than they might have otherwise. So work hard on the problems, and good luck on the Team Play!

Facts: Given real numbers a and b, the value of $\min(a, b)$ is equal to the smaller of a and b. If a and b happen to be equal, then $\min(a, b)$ equals this common value. Thus $\min(\pi, \sqrt{10}) = \pi$, while $\min(\frac{8}{5}, 1.6) = 1.6$.

Each term in an *arithmetic sequence* differs from the previous one by a fixed amount, as in the sequence 4, 7, 10, 13, ..., 100. To find their sum, count how many terms there are, compute the average value of the terms, and multiply these quantities. In our case there are $\frac{1}{3}(100 - 4) + 1 = 33$ terms, the average is $\frac{1}{2}(4 + 100) = 52$, for a sum of $33 \cdot 52 = 1716$.

Topics: iteration, sum of terms of an arithmetic sequence, induction, invariant of a process

Practice Problems

Let a_1 and b_1 be positive real numbers. Obtain a second pair of numbers by setting $a_2 = \min(a_1, b_1)$ and $b_2 = |a_1 - b_1|$. In other words, a_2 is the smaller of a_1 and b_1, while b_2 is the positive difference between a_1 and b_1. We then iterate this process by letting $a_3 = \min(a_2, b_2)$ and $b_3 = |a_2 - b_2|$, then $a_4 = \min(a_3, b_3)$ and $b_4 = |a_3 - b_3|$, and so forth. We say that this process *stabilizes* at step k if the pair (a_k, b_k) matches the next pair but differs from the previous pair.

1. Find the sum of the terms in the arithmetic sequence 5, 11, 17, 23, ..., 101.

2. Carry out the above process when $a_1 = 9$, $b_1 = 31$. At which step does the process stabilize? Now answer the same questions for $a_1 = 2\frac{2}{3}$, $b_1 = 3\frac{2}{3}$, and also for $a_1 = 1$, $b_1 = \sqrt{2}$.

3. Under what conditions will our process stabilize? (Recall that a_1, b_1 are positive real numbers.)

4. When the process does stabilize at step k, what is the significance of the value of b_k?

5. Let $a_1 = n$ and $b_1 = n^2 + 1$ for a positive integer n. At which step does the process stabilize? (Your answer will be in terms of n.)

6. Prove that it is possible to choose positive integers a_1 and b_1 in such a way that every value of a_k is a perfect square, and 2013 different squares appear in the process.

Hints and answers on the next page. $\Longrightarrow$

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Hints and Answers

1. There are $\frac{1}{6}(101-5)+1=17$ terms, with an average value of $\frac{1}{2}(5+101)=53$, so the sum of all the terms is $17\cdot 53=901$.

2. We find that the process stabilizes at step 11, since the tenth pair is $(1,0)$, while the eleventh and all subsequent pairs are $(0,1)$. Similarly, the next example stabilizes at step 8 when it reaches the pair $(0,\frac{1}{3})$, while the third example never stabilizes.

3. The process stabilizes exactly when the ratio a_1/b_1 is a rational number. Notice that a_1 and b_1 themselves might be irrational; for instance, try starting with the pair $a_1=2\sqrt{2}$ and $b_1=3\sqrt{2}$, which stabilizes at step 5.

4. As the examples in the previous parts suggest, the value of b_k is the "real GCD" of a_1 and b_1. More precisely, b_k is equal to the unique real number d for which a_1/d and b_1/d are relatively prime positive integers.

5. You should find that the process stabilizes at step $2n+2$. To check that you're on the right track, pair n is $(n,n+1)$ and pair $2n$ is $(1,1)$.

6. We will outline the solution. Returning to the very first question, we find an example of a sequence of pairs beginning with $(9,31)$ for which every first element is a perfect square. Imagine extending this sequence "upward" to include more pairs prior to $(9,31)$. By repeatedly adding 9 to 31 we eventually reach another perfect square, namely 49. At this point we can switch to an initial value of 49 by using the pair $(49,58)$ prior to $(49,9)$ and $(9,40)$. Now continually adding 49 to 58 we are again guaranteed to reach another perfect square, since these numbers are all congruent to 9 mod 49. In fact the next such square will be $(49-3)^2=2116$. We now switch to the pair $(2116,49)$, preceded by $(2116,2165)$. Continuing in this fashion we can ensure that we obtain arbitrarily many different perfect squares as values of a_k.