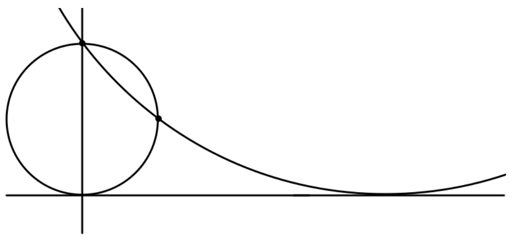
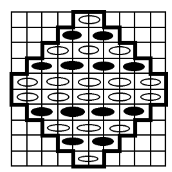
Answer Key	
1. 2	4. 40
2. 6	5. $F_{2014}F_{2011}$
3. 7	6. 768
	7. $\frac{1}{2}(\sqrt{3}+1)$

1. There are exactly **two** such circles, as illustrated by the diagram below. This is generally true—but can you find positions for the two given points for which there is only one such circle?



2. The sum is easier to evaluate if we first pair adjacent numbers as suggested in the diagram. Because of the location of the even and odd numbers, each pair covered by a white oval sums to $+1$, while each pair covered by a dark oval adds to -1. Hence the overall total is $18-12=\mathbf{6}$.



3. Here is a curious instance in which the straight line path does not give the minimal answer! To steer clear of as many points with integer coordinates as possible, follow a staircase path from $(1,1)$ to $(4,4)$. It is equally effective to head straight across to $(4,1)$ then up to $(4,4)$, or perhaps to go up to $(1,4)$ first before moving across to $(4,4)$. Any of these paths yields exactly **7** red points.

4. The sum of a sequence of consecutive positive integers is equal to the number of terms multiplied by the average of the first and last terms, which gives

$$14+15+16+\cdots+n=(n-14+1)\cdot\tfrac{1}{2}(n+14).$$

To obtain a power of 3, each separate factor must be a power of 3. Thus both $n-13$ and $\frac{1}{2}(n+14)$ must be one of 1, 3, 9, 27, ..., giving

$$n=14,\ 16,\ 22,\ 40,\ \ldots \qquad \text{and} \qquad n=-12,\ -8,\ 4,\ 40,\ \ldots$$

The smallest value of n for which both factors are powers of 3 is $n=\mathbf{40}$.

5. Observe that by substituting

$$3F_{2012}^2=3(F_{2011}+F_{2010})^2=3F_{2011}^2+6F_{2011}F_{2010}+3F_{2010}^2,$$

the given expression may be rewritten as

$$3F_{2012}^2-4F_{2011}F_{2010}-3F_{2010}^2=3F_{2011}^2+2F_{2011}F_{2010},$$

which factors as $F_{2011}(3F_{2011}+2F_{2010})$. To handle the second factor, we compute

$$\begin{aligned}3F_{2011}+2F_{2010} &= 2F_{2012}+F_{2011}\\ &= F_{2013}+F_{2012}\\ &= F_{2014},\end{aligned}$$

by definition of Fibonacci numbers. Hence the entire expression reduces to $\boldsymbol{F_{2014}F_{2011}}$.

6. One shortcut to handling the count is to observe that reversing the colors of every square within a particular column does not affect whether or not a certain coloring satisfies the given conditions. This means that we can assume that the top row contains only white squares, find a legitimate coloring, then reverse the squares within any subset of columns, which can be done in $2^4=16$ ways.

So for now we assume that the top row contains only white squares. The next row must contain a pair of black squares, which can be placed in $\binom{4}{2}=6$ ways. The third row must also contain a pair of black squares, one that overlaps with a previous black square and one that doesn't, which can be done in 4 ways. Regardless of the positions of the black squares thus far, the final row can be colored in exactly 2 ways. (Why?) Hence the overall total is $(16)(6)(4)(2)=\mathbf{768}$.

7. This curious fraction is self-similar in the following manner. Doubling the entire expression gives

$$2+\frac{4}{2}\cdot\cfrac{1}{2+\cfrac{4}{2^2+\cfrac{4^2}{2^3+\cfrac{4^3}{2^4+\cdots}}}}=2+\cfrac{4}{2^2+\frac{4}{2}\cdot\cfrac{4}{2^2+\cfrac{4^2}{2^3+\cfrac{4^3}{2^4+\cdots}}}}=$$

$$2+\cfrac{4}{2^2+\cfrac{4^2}{2^3+\frac{4}{2}\cdot\cfrac{4^2}{2^3+\cfrac{4^3}{2^4+\cdots}}}}=2+\cfrac{4}{2^2+\cfrac{4^2}{2^3+\cfrac{4^3}{2^4+\frac{4}{2}\cdot\cfrac{4^3}{2^4+\cdots}}}}=\cdots$$

Continuing in this fashion, it becomes clear that if we call the value of the entire expression V, then the quantity $2V$ appears as the "main denominator" of the original expression. In other words,

$$V=1+\frac{1}{2V}.$$

It is routine to solve the resulting quadratic to find that the positive value of V is $\boldsymbol{\frac{1}{2}(\sqrt{3}+1)}$.

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