ANSWER KEY		4.	2016
1.	13	5.	431256
2.	6	6.	343
3.	$\{3, -2, -2, -2\}^*$	7.	$21\sqrt{123}$

*See below for an alternate acceptable answer.

1. The first answer to spring to mind is $x = 21$, which clearly satisfies the equation. However, there is another solution, which can be found by observing that when $x \le 14$ it is the case that $|x-20| + |x-14| = 8$ reduces to $(20-x)+(14-x) = 8$. Solving gives $34-2x = 8$, or $26 = 2x$, so $x = \mathbf{13}$, the smallest solution.

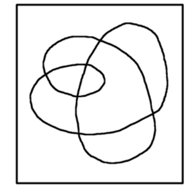
2. Consider any particular red marker. In order for it to return to its starting position it must move at least twice, and in order to have the black side up it must move an odd number of times. Therefore each marker must move at least three times, for a total of twelve moves among all the markers. Each swap moves two markers, so we need at least six swaps. It is possible to accomplish the task with exactly six swaps (the reader should figure out how!), so the answer is **6**.

3. Problems involving sums of cubes are an intriguing but somewhat mysterious branch of number theory. For instance, it is likely that there are infinitely many sets of four integers whose sum is -3 and whose sum of cubes is 3. The smallest and most readily found such set is $\{\mathbf{3}, \mathbf{-2}, \mathbf{-2}, \mathbf{-2}\}$, since the sum of these integers is clearly -3 and

$$3^3 + (-2)^3 + (-2)^3 + (-2)^3 = 27 - 8 - 8 - 8 = 3.$$

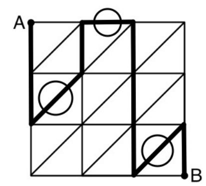
However, there are more possible answers, including $\{7, 1, -5, -6\}$ and $\{10, 3, -8, -8\}$. The interplay between sums and cubes is marvelous!

4. The doodle shown crosses over itself six times and leads to eight regions. Some experimentation reveals that a doodle crossing over itself five times gives seven regions, and in general the number of regions is always two more than the total number of self-intersections. This can be explained by using Euler's Formula, which states that $V - E + F = 2$. In our situation V is the number of vertices (doodle crossings), E is the number of edges (pieces into which the crossings chop up the doodle), and F is the number of faces (regions). We encounter each crossing twice as we move around the doodle, so $E = 2V$, leading to $F = V + 2$, as claimed.

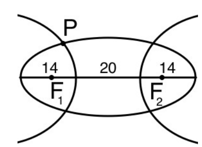


5. Our six-digit number cannot end with a 5 (since it is divisible by 6), but upon deleting the 6 it *must* end with 5 (in order to be divisible by 5). Therefore our six-digit number must have the form $****56$. The largest possible number of this form is 432156, which doesn't quite work since 4321 is not a multiple of 4. However, **431256** does work.

6. The key is to realize that every path from A to B is uniquely determined by the three segments used to traverse from one vertical column to the next. (These segments are marked with a circle in the diagram.) Once these three segments are chosen, the rest of the path is automatically determined. There are seven such segments between each pair of columns, for a total of $7^3 = \mathbf{343}$ paths.



7. The given information is shown in the diagram at right; we have omitted the labels Q, R, S and T to avoid cluttering the diagram. First consider the ellipse. The major axis has length $14+20+14 = 48$, hence according to the geometric definition of an ellipse we know $PF_1 + PF_2 = 48$. Meanwhile, the distance between the vertices of the hyperbola is 20, so by the geometric definition we deduce that $PF_2 - PF_1 = 20$. Solving, we see that $PF_1 = 14$ while $PF_2 = 34$. Since $\triangle PF_1F_2$ is isosceles we also have $F_1F_2 = 34$. Finally, by the Pythagorean Theorem the height from F_2 to PF_1 has length $\sqrt{34^2 - 7^2} = 3\sqrt{123}$, so $area(PF_1F_2) = \frac{1}{2}(14)(3\sqrt{123}) = \mathbf{21\sqrt{123}}$.



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