February 2012

The Mandelbrot Team Play

Round Two Test

Time Limit: 60 minutes

Facts: *Vieta's formulas* provide a means of computing symmetric expressions involving the roots of a polynomial. More precisely, the polynomial $c_0x^n + c_1x^{n-1} + c_2x^{n-2} + \cdots + c_{n-1}x + c_n$ has n roots. According to Vieta's formulas, the sum of these roots is $-c_1/c_0$, the sum of products of pairs of distinct roots is c_2/c_0, and the product of the roots is $(-1)^n c_n/c_0$.

Setup: Let $n \geq 2$ be a positive integer, and consider the equation

$$\frac{1}{1-x} + \frac{1}{2-x} + \frac{1}{3-x} + \cdots + \frac{1}{n-x} = 0. \qquad (*)$$

It turns out that this equation is satisfied by exactly $n-1$ real numbers. Let us call these solutions $a_1, a_2, \ldots, a_{n-1}$, in increasing order. In the questions below we will investigate various properties of these solutions.

Problems

Part i: (4 points) Write out equation $(*)$ in the case $n = 3$ and create an accurate sketch of the graph of the left-hand side, indicating asymptotes with dashed lines. Based on this sketch, where are the two solutions located? Then solve the equation, giving exact answers.

Part ii: (4 points) How are the solutions of the equation

$$\frac{1}{1-x} + \frac{2}{2-x} + \frac{3}{3-x} + \cdots + \frac{n}{n-x} = n$$

related to the solutions of equation $(*)$?

Part iii: (5 points) Explain why $a_k + a_{n-k} = n + 1$ for all $1 \leq k \leq n-1$. Capitalize on this fact to find an expression (in terms of n) for the sum of all the solutions.

Part iv: (5 points) Prove that the difference between any two solutions is greater than 1.

Part v: (5 points) Let $A = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ and $B = 1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$. Demonstrate that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{n-1}} = A - \frac{B}{A}.$$

Part vi: (5 points) Let $A = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ as before. Prove that the smallest root a_1 falls in the range $1 < a_1 < 1 + \frac{1}{A}$.