Team Play Topics

Round One

The first section of the Round One Mandelbrot Team Play is reproduced below. A list of topics and practice problems are also provided to aid in preparation. Note that these problems are not meant to serve as a precise indicator of the problems that will appear on the contest. However, students who understand how to solve them should be able to make significantly more progress than they might have otherwise. So work hard on the problems, and good luck on the Team Play!

Facts: Every positive integer can be written uniquely as a sum of distinct powers of 2. For example,

$$57 = 32 + 16 + 8 + 1 = 2^5 + 2^4 + 2^3 + 2^0.$$

We write this equality more compactly as $57 = 111001_2$, using 1's and 0's to indicate which powers of 2 to include or exclude from the sum, in order from largest to smallest. This is known as the *binary representation* of n. In a similar manner we find that $19 = 10011_2$, $31 = 11111_2$ and $32 = 100000_2$. It also makes sense to write $0 = 0_2$. Finally, recall that $\binom{r}{2}$ counts the number of ways to choose two objects from among a set of r objects.

Topics: Binary representation, binary arithmetic, basic counting techniques, $\binom{n}{2}$, base three

Practice Problems

1. Determine the binary representations for 42, 67 and 2011.

2. Find positive integers m and n that each have three 1's in their binary representation, but such that $m + n$ has only a single 1 in its binary representation.

3. Add the binary numbers $10111_2 + 110110_2$. Perform the addition directly in binary, noting that $1 + 1 = 10_2$, $1 + 1 + 1 = 11_2$, carrying 1's where necessary, and writing your answer in binary.

4. Given $r \geq 1$, how many digits does $2^r - 1$ have in binary? How many integers from 0 to $2^r - 1$ (inclusive) have exactly two 1's in their binary representation?

5. How many integers from 0 to $2^r - 1$ end with a pair of 1's? How many integers from 0 to $2^r - 1$ begin with a pair of 1's?

6. Show that for any positive integers m and n, the number of 1's in the binary representation of $m + n$ never exceeds the combined total of 1's in the binary representations of m and n.

Hints and answers on the next page. $\Longrightarrow$

Team Play Topics

Hints and Answers

1. We find that $42 = 101010_2$, $67 = 1000011_2$ and $2011 = 11111011011_2$. The latter is found most efficiently by noting that $2011 = 2047 - 36 = 11111111111_2 - 100100_2$.

2. There are many possible answers; for instance, take $m = 7$ and $n = 25$.

3. The answer is 1001101_2; the computation is shown at right.

$$\begin{array}{cccccccc} & {\scriptstyle 1} & {\scriptstyle 1} & & {\scriptstyle 1} & {\scriptstyle 1} & & \\ & & & 1 & 0 & 1 & 1 & 1 \\ + & & 1 & 1 & 0 & 1 & 1 & 0 \\ \hline & 1 & 0 & 0 & 1 & 1 & 0 & 1 \end{array}$$

4. The number $2^r - 1$ consists of r 1's in binary. Of the 2^r integers from 0 to $2^r - 1$, exactly $\binom{r}{2}$ of them have two 1's in their binary representation, since we may choose any 2 of the r digits for the 1's, then place 0's in the remaining positions.

5. There are 2^{r-2} integers from 0 to $2^r - 1$ that end with a pair of 1's, since each of the remaining $r - 2$ digits may be either a 0 or 1. Similarly, there are

$$2^{r-2} + 2^{r-3} + \cdots + 2 + 1 = 2^{r-1} - 1$$

such integers that begin with a pair of 1's. (Observe that this time we must keep track of how many digits there are in our number; anywhere from r digits down to 2 digits.)

6. Write each of m and n as a sum of distinct powers of 2. Adding these sums gives a way to write $m + n$ as a sum of powers of 2, possibly with repeats. Combining pairs like $2^k + 2^k = 2^{k+1}$, we eventually reach a way to write $m + n$ as a sum of *distinct* powers of 2. In the process the total number of powers of 2 employed can only remain the same or decrease, as claimed.