Team Play Solutions

Part i: Rows six to ten of the Lucas triangle are displayed below.

1	7	20	30	25	11	2				
1	8	27	50	55	36	13	2			
1	9	35	77	105	91	49	15	2		
1	10	44	112	182	196	140	64	17	2	
1	11	54	156	294	378	336	204	81	19	2

According to the Setup section, there should be $L(5,2) = 14$ tilings of a bracelet of length 7 using two dominoes and three squares:

$$ddsss,\ Ddsss,\ dsdss,\ Dsdss,\ dssds,\ Dssds,\ dsssd,\ Dsssd,$$
$$sddss,\ sdsds,\ sdssd,\ ssdds,\ ssdsd,\ sssdd.$$

Part ii: We know that $L(n,0)$ counts the number of tilings of a bracelet of length n using no dominoes and n squares. Clearly there is only one such tiling, namely $ss\cdots s$, so $L(n,0) = 1$. On the other hand, $L(n,n)$ counts tilings of a bracelet of length $2n$ using n dominoes and no squares. Now there are two such tilings, either $dd\cdots d$ or $Dd\cdots d$, so $L(n,n) = 2$.

Next consider $L(n,1)$, counting tilings of a bracelet of length $n+1$ using one domino and $n-1$ squares. Imagine first creating a row of $n-1$ squares, then inserting the domino. There are n possible positions, from the first spot to the last, giving $dss\cdots s$, $sds\cdots s$, up to $ss\cdots sd$. We obtain one more tiling by capitalizing the initial d in a single case as $Dss\cdots s$, for a total of $n+1$ tilings, thus $L(n,1) = n+1$.

Finally, $L(n,n-1)$ counts tilings of a bracelet of length $2n-1$ using $n-1$ dominoes and one square. As just explained, there are n ways to insert a square into a row of $n-1$ dominoes. But this time we can capitalize the initial d in all but one case, leading to a total of $2n-1$ tilings, so we conclude that $L(n,n-1) = 2n-1$.

Part iii: The entries in the Lucas triangle corresponding to $L(n,n-2)$ are all perfect squares, so we guess that $L(n,n-2) = (n-1)^2$. To see that this makes sense, recall that $L(n,n-2)$ counts the number of ways to tile a bracelet of length $2n-2$ with $n-2$ dominoes and two squares. At this stage there is a clever way to avoid dealing with the clasp or separately handling the cases of tilings with an initial D. There are $2n-2$ ways to put the first square onto any section of the bracelet, but then only $n-1$ ways to place the second square, since we must leave an even number of sections between the squares so that the dominoes fit. (Verify this.) Dividing by 2, since each placement of the two squares is counted twice in the process, we obtain $L(n,n-2) = \frac{1}{2}(2n-2)(n-1) = (n-1)^2$.

One naturally wonders whether there is a corresponding formula for $L(n,2)$. In fact, we have $L(n,2) = \frac{1}{2}(n+2)(n-1)$, which may be proved in a very similar manner to the above. The reader is invited to discover the explanation before reading further.

Part iv: Here is a short, sweet combinatorial argument. We know from the Facts section that there are L_n ways to tile a bracelet of length n. This may be accomplished by using n squares, or $n-2$ squares and one domino, or $n-4$ squares and two dominoes, etc. By the Setup section, these separate cases (using different sets of tiles) lead to $L(n,0)$ tilings, or $L(n-1,1)$ tilings, or $L(n-2,2)$ tilings, etc. But these are exactly the numbers along a line of slope 1 through $L(n,0)$ within the Lucas triangle. Therefore

$$L(n,0) + L(n-1,1) + L(n-2,2) + \cdots = L_n.$$

With this in mind, the top diagonal through $L(0,0)$ should add up to $L_0 = 2$, suggesting that it is proper to define $L(0,0) = 2$.

Part v: The first several row sums are 3, 6, 12, 24, 48, $\ldots$, leading us to conjecture that the sum of the entries of row n is equal to $3(2^{n-1})$. This formula becomes delightfully transparent using the fact that for a fixed value of n, $L(n,k)$ counts tilings involving a total of n tiles; k of them dominoes and the rest squares. To create a tiling using precisely n tiles we may begin with either s, d or D, giving three choices. For each subsequent tile we must use either s or d, so there are two choices for each of the remaining $n-1$ positions, bringing the number of tilings to

$$3 \cdot 2 \cdot 2 \cdots 2 = 3(2^{n-1}).$$

Part vi: For larger primes this behavior is even more striking. For instance, the diagonal with slope 1 through $L(11,0)$ includes the numbers 1, 11, 44, 77, 55, 11. To understand this phenomenon, recall from above that the values along the diagonal through $L(p,0)$—but not including $L(p,0)$—have the form $L(p-k,k)$ for $1 \leq k < \frac{1}{2}p$, so these values count the number of ways to tile a bracelet of length p with some number of squares and at least one domino. Now consider any particular such tiling, say using k dominoes. By rotating the tiles one section at a time around the bracelet we obtain a total of p tilings having the same "structure," including the original one.

We claim that all these tilings are distinct. (Observe that this must have something to do with the fact that p is prime, since the tiling *sdsdsdsd* of a bracelet of length 12 does *not* yield a new tiling when the tiles are rotated by three sections in either direction.) For suppose we rotate the original tiles by b sections counterclockwise, where $0 < b < p$, and track the motion of any particular domino. If the new tiling matches the original one, then the tile b sections away from this domino must also be a domino. By the same reasoning, the tile b sections away from that domino (i.e. $2b$ sections around from the original domino) must also be a domino, and so forth. Because b is relatively prime to p, this process will not end until we have rotated the original domino p times, at which point it returns to its starting position. We conclude that the tiling would involve p dominoes, which is impossible, since there are only p sections in the bracelet. Hence all p tilings are distinct.

Therefore, given any admissible value of k, the tilings of a bracelet of length p with k dominoes and the rest squares can be sorted into groups, with p tilings in each group. It follows that $L(p-k,k)$ is divisible by p.

We are grateful to Art Benjamin for his permission to create this set of Team Play problems, which are based on the material and methods described in his paper The Lucas Triangle Recounted, available for free download at Art Benjamin's website.

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Round Two Solutions

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