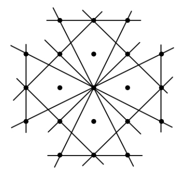
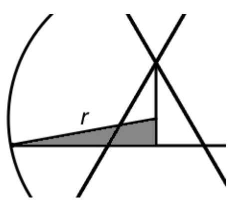
Answer Key	
1. 14	4. 236
2. 273	5. 3/4
3. $7\pi/3$	6. 17/28
	7. 230300



1. The illustration to the right shows all **14** lines that pass through exactly three points in the grid.

2. One quickly realizes that no number from 250 to 259 can work, since twice such a number begins with a 5, repeating that digit. Even numbers are also not likely to work, since they produce a lot of even digits. Checking the other possibilities leads one to discover before long that the first three multiples of **273** are 273, 546, 819, which contain each of the digits from 1 to 9 exactly once. (Note that 192, 384, 576 is another such triple.)



3. Perhaps the most direct approach is to use the Pythagorean Theorem to compute the radius of the circle. Focusing on the part of the diagram shown, the altitude of the middle equilateral triangle is $\sqrt{3}/2$ using 30°–60°–90° triangle facts. The center of the triangle (and hence of the entire circle) is one-third of the way up this altitude, which is a distance of $\sqrt{3}/6$. Furthermore, half the horizontal chord has length $3/2$. This gives the two legs of the shaded right triangle, so the hypotenuse is

$$r = \sqrt{\left(\frac{\sqrt{3}}{6}\right)^2 + \left(\frac{3}{2}\right)^2} = \sqrt{\frac{1}{12} + \frac{9}{4}} = \sqrt{\frac{7}{3}}.$$

Hence the area of the circle is $\pi r^2 = \mathbf{7\pi/3}$.

4. Since our positive integer requires four digits in base five it must be at least 125, since $125 = 1000_5$. But it has only seven binary digits, so it can be at most 127, since $127 = 1111111_2$. Together this limits the possibilities to either 125, 126 or 127. But we have just seen that 127 is a palindrome in base two, while $126 = 1001_5$ which is also a palindrome. Hence the answer is 125, which is written as **236** in base seven.

5. To solve for x we multiply the first equation by $\log_5 21$ and the second equation by $\log_5 13$, then subtract. The y terms now cancel, leaving

$$(\log_5 21)(\log_{21} 48)x - (\log_5 13)(\log_{13} 3)x = (\log_5 21)(\log_{21} 56) - (\log_5 13)(\log_{13} 7).$$

Using the fact that $(\log_a b)(\log_b c) = \log_a c$, the above equation becomes

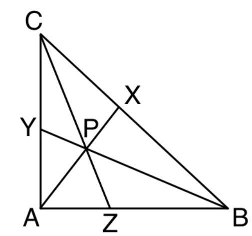
$$(\log_5 48 - \log_5 3)x = \log_5 56 - \log_5 7.$$

Now we employ the subtraction law $\log_a b - \log_a c = \log_a(\frac{b}{c})$ to obtain

$$(\log_5 16)x = \log_5 8 \quad\Longrightarrow\quad x = \frac{\log_5 8}{\log_5 16} = \frac{3\log_5 2}{4\log_5 2} = \mathbf{\frac{3}{4}}.$$

(Note that answers involving $\log_5$ are not in simplest form, so should not receive credit.)

6. Here is a powerful approach worth understanding. Perform an affine transformation (also called a parallel projection) that takes points A, B and C to the vertices of an isosceles right triangle with sides of length 45. (This will make computations come out more neatly later.) Since ratios are preserved under such a transformation, we obtain the diagram shown at right in which $PY/PB = 2/7$. But since $AB = 45$, we can now deduce that the x-coordinate of P is $\frac{2}{9}(45) = 10$, where A is the origin of a coordinate system. Similarly, we find that the y-coordinate of P is $\frac{2}{5}(45) = 18$. Therefore the equation of line AP is $y = \frac{9}{5}x$, which will intersect line BC with equation $x + y = 45$ at point $X\left(\frac{5}{14}(45), \frac{9}{14}(45)\right)$. Finally, we compute the ratio PX/PA via the x-coordinates as



$$\frac{\frac{225}{14} - 10}{10} = \frac{1}{10}\left(\frac{85}{14}\right) = \mathbf{\frac{17}{28}}.$$

7. In general, we claim that the number of ways for the cashier to hand \$$n$ to the customer having five distinguishable bills in each of the denominations \$1, \$2, \$4, \$8, \$16, ... is exactly $\binom{n+4}{4}$. One might come to this conclusion by testing small values of n and noticing a pattern, or perhaps by utilizing the generating function

$$(1+x)^5(1+x^2)^5(1+x^4)^5(1+x^8)^5\cdots = \frac{1}{(1-x)^5} = \sum_{n=0}^{\infty}\binom{n+4}{4}x^n.$$

Regardless, since the cashier needs to make change for \$46, we obtain the answer as

$$\binom{50}{4} = \frac{50\cdot 49\cdot 48\cdot 47}{4\cdot 3\cdot 2\cdot 1} = \mathbf{230300}.$$

Our purpose here is to present a beautiful demonstration of this fact. To make the bills more obviously distinct, color the \$1 bills red, blue, green, orange, and purple, then do the same for the \$2 bills, and so forth. Now imagine a row of 50 white chips, choose any four of them, and color those four chips black. Clearly this can be done in $\binom{50}{4}$ ways. We now claim that each such selection corresponds to exactly one way to make change for \$46, and vice-versa, which would complete the proof.

The black chips split the remaining 46 white chips into five piles (some of which may be empty, if two black chips happen to be adjacent) in a row. Color the first pile red, the next blue, then green, orange, and purple. Then further split the red chips into smaller piles via the binary representation of the size of the pile. (So a pile of 13 red chips would be subdivided into piles of size 8, 4 and 1.) Do the same for the other four piles. Finally, use the pile sizes to indicate the bills to be used in making change; i.e. a pile of 8 red chips means use a red \$8 bill. In this manner we obtain each possible way of making change, and we're done.

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⋆ National Level ⋆

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