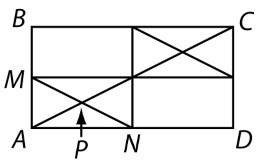
Answer Key	
1. $\frac{1}{3}$	4. 91
2. 9	5. $\frac{11}{4}$
3. 8	6. 196
	7. $x^2 + 1774x + 235$

1. Segment $\overline{MN}$ and diagonal $\overline{AC}$ are drawn within rectangle $ABCD$, intersecting at P. Now imagine drawing the midpoints of the other two sides and connecting the various midpoints as shown. Clearly we will create four smaller rectangles, and the various segments will divide diagonal $\overline{AC}$ into four equal parts. Since $\overline{AP}$ is one of these parts while $\overline{PC}$ covers the remaining three, we will have $AP/PC = \mathbf{1/3}$.



2. Since we are not told which girl is twice as old as another we consider three cases. To begin, suppose that the ages of the girls are currently x, $2x$ and y. Notice that it will take longer for the oldest girl to become twice the age of the youngest as opposed to the middle girl. (Why?) So in one year their ages will be $x+1$, $2x+1$ and $y+1$ and we must have $y+1 = 2(2x+1)$. Four years later their ages will be $x+5$, $2x+5$ and $y+5$ and we will have $y+5 = 2(x+5)$. Solving these two equations for x and y yields $x = 2$, $y = 9$. If the ages of the girls were y, x and $2x$ instead, then the resulting equations give $x = 4$, $y = 1.5$, which doesn't work. Furthermore, the case x, y, $2x$ is impossible, according to the above note. Hence the oldest girl is currently **9**.

3. We first factor $84,000 = 2^5 \cdot 3 \cdot 5^3 \cdot 7$. There are many ways to write 84,000 as a product mn, but if m and n are to be relatively prime, then all five factors of 2 must present in either m or n, so that m and n don't have a common factor of 2. For the same reason all three factors of 5 must occur in either m or n. This leads to **8** distinct products:

$$(1)(2^5 \cdot 3 \cdot 5^3 \cdot 7),\ (2^5)(3 \cdot 5^3 \cdot 7),\ (3)(2^5 \cdot 5^3 \cdot 7),\ (5^3)(2^5 \cdot 3 \cdot 7),$$

$$(7)(2^5 \cdot 3 \cdot 5^3),\ (2^5 \cdot 3)(5^3 \cdot 7),\ (2^5 \cdot 5^3)(3 \cdot 7),\ \text{and } (2^5 \cdot 7)(3 \cdot 5^3).$$

4. If we begin computing values of $g(n)$ for $n = 1, 2, 3, \ldots$ it seems at first as though no output will be repeated. This is in fact the case for two-digit numbers: let $m = 10a + b$ and $n = 10c + d$ be two digit numbers; then $g(m) = (10a + b) + a + b = 11a + 2b$ and $g(n) = 11c + 2d$. If it were the case $g(m) = g(n)$ then we would have $11(a - c) = 2(d - b)$. But digits b and d can't differ by a multiple of 11 unless $b = d$, which means that $a = c$ also, which forces m and n to be the same number. However, as soon as we reach three digit numbers there is a repeated value, since $g(91) = 91 + 9 + 1 = 101$ and $g(100) = 100 + 1 + 0 + 0 = 101$ also. Hence **91** is the smallest value of n that gives a repeated output.

5. Using the standard double-angle formulas $\cos(2\theta) = \cos^2\theta - \sin^2\theta$ and $\sin(2\theta) = 2\sin\theta\cos\theta$ we may rewrite the given expression as

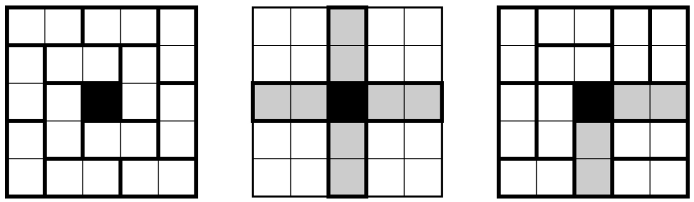
$$\frac{41}{16} = 4\cos(2\theta) + 3\sin(2\theta) = 4\cos^2\theta + 6\sin\theta\cos\theta - 4\sin^2\theta.$$

This expression vaguely resembles $(3\cos\theta + \sin\theta)^2$; it has the correct general form and the correct cross-term. If we add $5\cos^2\theta + 5\sin^2\theta = 5$ to both sides then we obtain precisely that perfect square:

$$5 + \frac{41}{16} = 9\cos^2\theta + 6\sin\theta\cos\theta + \sin^2\theta = (3\cos\theta + \sin\theta)^2.$$

Taking square roots gives $3\cos\theta + \sin\theta = \sqrt{121/16} = \mathbf{11/4}$. (We choose the positive square root since θ is in the first quadrant.)

6. To begin, notice that there are two configurations in which the dominoes are "staggered" in concentric rings, one of which is shown at left below. All other configurations involve dominoes which pair up into 2×2 blocks, such as the example given in the problem. For these cases we choose to organize our count according to the number of "pivot dominoes" that appear in our tiling. (The pivot dominoes are the ones shaded in the center diagram below.)



For example, a tiling involving two adjacent pivot dominoes, such as the configuration on the right above, can happen in exactly 48 ways: there are 4 ways to choose a pair of adjacent pivot dominoes, 2 ways to fill the 2×2 block they enclose, 3 ways to cover the other two pivot domino locations so that a pivot domino does not appear there, and 2 ways to complete the tiling in each case.

There are 16 ways to include all four pivot dominoes, 64 ways to use three pivot dominoes, 32 ways to include 2 nonadjacent pivot dominoes, 32 ways to use only one pivot domino, and only 2 ways to avoid all pivot dominoes. (Details are left to the reader to supply.) This gives a total of **196** tilings in total.

7. This problem involves a fairly stiff dose of algebra, but otherwise is not as difficult as it looks. To begin, $f(x) + x = x^2 + (a+1)x + b$, so

$$\begin{aligned} f(f(x)+x) &= (x^2 + (a+1)x + b)^2 + a(x^2 + (a+1)x + b) + b \\ &= x^4 + (2a+2)x^3 + (a^2 + 3a + 2b + 1)x^2 + \\ &\qquad (a^2 + 2ab + a + 2b)x + (b^2 + ab + b) \\ &= (x^2 + ax + b)(x^2 + (a+2)x + a + b + 1). \end{aligned}$$

Remarkably, the resulting expression factors as shown. Therefore dividing by $f(x)$ leaves $x^2 + (a+2)x + a + b + 1$. We need $a + 2 = 1776$ and $a + b + 1 = 2010$, which means we should take $a = 1774$ and $b = 235$, giving $f(x) = \mathbf{x^2 + 1774x + 235}$.

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⋆ National Level ⋆

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Round Five Solutions

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