Answer Key

1. 13
2. 6
3. $\{3, -2, -2, -2\}^*$
4. 431256
5. 343
6. $6\sqrt{2}$
7. 20000

*See below for an alternate acceptable answer.

1. The first answer to spring to mind is $x = 21$, which clearly satisfies the equation. However, there is another solution, which can be found by observing that when $x \le 14$ it is the case that $|x - 20| + |x - 14| = 8$ reduces to $(20 - x) + (14 - x) = 8$. Solving gives $34 - 2x = 8$, or $26 = 2x$, so $x = \mathbf{13}$, the smallest solution.

2. Consider any particular red marker. In order for it to return to its starting position it must move at least twice, and in order to have the black side up it must move an odd number of times. Therefore each marker must move at least three times, for a total of twelve moves among all the markers. Each swap moves two markers, so we need at least six swaps. It is possible to accomplish the task with exactly six swaps (the reader should figure out how!), so the answer is **6**.

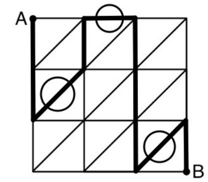
3. Problems involving sums of cubes are an intriguing but somewhat mysterious branch of number theory. For instance, it is likely that there are infinitely many sets of four integers whose sum is -3 and whose sum of cubes is 3. The smallest and most readily found such set is $\{\mathbf{3}, -\mathbf{2}, -\mathbf{2}, -\mathbf{2}\}$, since the sum of these integers is clearly -3 and

$$3^3 + (-2)^3 + (-2)^3 + (-2)^3 = 27 - 8 - 8 - 8 = 3.$$

However, there are more possible answers, including $\{7, 1, -5, -6\}$ and $\{10, 3, -8, -8\}$. Isn't it marvelous that the pairs of numbers $10+3$, $8+8$ and $10^3 + 3^3$, $8^3 + 8^3$ both differ by 3?

4. Our six-digit number cannot end with a 5 (since it is divisible by 6), but upon deleting the 6 it *must* end with 5 (in order to be divisible by 5). Therefore our six-digit number must have the form $****56$. The largest possible number of this form is 432156, which doesn't quite work since 4321 is not a multiple of 4. However, **431256** does work. and hence is the largest possible answer.

5. The key is to realize that every path from A to B is uniquely determined by the three segments used to traverse from one vertical column to the next. (These segments are marked with a circle in the diagram.) Once these three segments are chosen, the rest of the path is automatically determined. There are seven such segments between each pair of columns, for a total of $7^3 = \mathbf{343}$ paths.



6. It is well-known that a quadrilateral with side lengths a, b, c and d (in that order) has an inscribed circle if and only if $a + c = b + d$, which explains the claim made in the statement of the problem. It is also a standard fact that in such a figure $K = rs$, where K is the area, r is the radius of the inscribed circle, and s is the semi-perimeter. Since $s = \frac{1}{2}(20 + 14 + 15 + 21) = 35$ is fixed, r is maximized exactly when K is. Brahmagupta's formula states that

$$K = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos\varphi},$$

where φ is the average of a pair of opposite angles. (It doesn't matter which pair.) Clearly this is maximized when $\cos\varphi = 0$, i.e. when the quadrilateral is cyclic. In our case this leads to a maximal area of $K = \sqrt{(15)(21)(20)(14)} = 210\sqrt{2}$, meaning $r = 210\sqrt{2}/35 = \mathbf{6\sqrt{2}}$.

7. To determine $a_{T_{100}}$ we first try smaller subscripts and note that

$$\begin{aligned} a_{T_1} &= a_1 = 2 = T_1 + 1, \\ a_{T_2} &= a_3 = 5 = T_2 + 2, \\ a_{T_3} &= a_6 = 9 = T_3 + 3, \\ a_{T_4} &= a_{10} = 14 = T_4 + 4, \end{aligned}$$

and so forth. It appears that $a_{T_k} = T_k + k$, which makes sense because exactly k positive integers are omitted among the first T_k terms in our sequence. Therefore $b_1 = T_{100} + 100$.

We next claim that in general $b_k = T_{99+k} + 100$. For observe that b_2 is term number $b_1 = T_{100} + 100$ in the sequence, just before term number T_{101}, which we know is equal to $T_{101} + 101$ by the above; thus $b_2 = T_{101}+100$. Inductively, b_{k+1} is term number $b_k = T_{99+k}+100$, just k terms prior to term number $T_{99+(k+1)}$, which equals $T_{99+(k+1)} + 100 + k$. Therefore $b_{k+1} = T_{99+(k+1)} + 100$, as desired. (One verifies that in backing up these k terms we don't pass over any triangular numbers, which are omitted from our sequence.) In summary, we have

$$b_{100} = T_{199} + 100 = \tfrac{1}{2}(199)(200) + 100 = \mathbf{20000}.$$

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