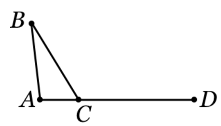
Answer Key

1. 7214563	4. 459
2. 2	5. -19
3. $5/12$	6. 23
	7. $\sqrt{5} - 2$

1. Finding the solution to this digit puzzle is mainly an exercise in organized trial and error. For instance, the digit 7 must be followed by a 2, unless it is the final digit. Similarly the 1 must be followed by the 4, if anything. Continuing in this manner one discovers the string of digits **7214563** before long. To our delight, this is the unique answer.



2. Because $AD = 8$ while $CD = 6$, it is impossible for point A to be any closer than **2** units to point C. Think of placing D at the center of the diagram and letting C circle around 6 units away, while A circles around 8 units away. Then A and C will be nearest when they both line up on one side of D, as shown in the diagram.

3. We can obtain a multiple of 6 in two distinct ways: either the coin comes up heads and the die shows a 3 or 6, or else the coin comes up tails and the die shows a 2, 4 or 6. The probability of the former is $(\frac{1}{2})(\frac{1}{3})$ while the probability of the latter is $(\frac{1}{2})(\frac{1}{2})$. Hence the overall probability is $\frac{1}{6} + \frac{1}{4} = \mathbf{\frac{5}{12}}$.

4. To build a fortunate number select primes $p, q, r \geq 5$, then add 4 to some multiple of pqr. Clearly this number leaves a remainder of 4 when divided by either p, q or r. For instance, the smallest fortunate number is $(5)(7)(11) + 4 = 389$. Unfortunately, it is not a multiple of 3. Another fortunate number is $2(5)(7)(11) + 4 = 774$, which is a multiple of 3. Unfortunately, it is not the smallest such number! We can do better using another triplet of primes; namely $(5)(7)(13) + 4 = \mathbf{459}$, giving the smallest fortunate multiple of 3.

5. The equation $f(x) = x$ can be rearranged to give $x^2 + 19x + 17 = 0$, which is a quadratic having two real roots r and s with sum $r + s = -19$ and product $rs = 17$. Since r is required to be a root of the second equation as well, we must have

$$a - f(a - f(r)) = r \quad \Longrightarrow \quad f(a - r) = a - r,$$

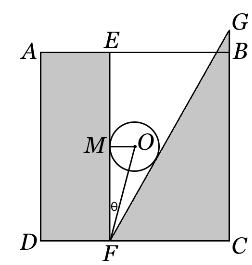
where we used the fact that $f(r) = r$ since r is a solution to $f(x) = x$. In other words, $a - r$ must also be a root of $f(x) = x$, which means that either $a - r = r$ or $a - r = s$. In the first case we have $a = 2r$, while the second case leads to $a = r + s = -19$. Using similar reasoning, the fact that s must also be a root of the second equation implies that either $a = 2s$ or $a = -19$. The only value of a for which both r and s are solutions to the second equation is $a = \mathbf{-19}$.

6. We observe that since b^{47} is divisible by neither 2 nor 5, then b is not either, so b is relatively prime to 100. It follows that $b^{40} \equiv 1 \bmod 100$, according to Euler's extension to Fermat's Little Theorem. Therefore we may deduce that $b^7 \equiv 47 \bmod 100$.

From here the way forward is less clear, unless you happen to know the *effective* way to apply Euler's extension. It is common knowledge that $b^2 \equiv 1 \bmod 4$ when b is odd, hence $b^{20} \equiv 1 \bmod 4$ as well. And by Euler's extension we know $b^{20} \equiv 1 \bmod 25$. Putting these together, we may conclude that $b^{20} \equiv 1 \bmod 100$, which is a just enough of an improvement over the b^{40} result to finish the problem. We compute

$$b^{14} \equiv (47)(47) \equiv 9 \bmod 100 \quad \Longrightarrow \quad b^{21} \equiv (47)(9) \equiv 23 \bmod 100.$$

Now using $b^{20} \equiv 1 \bmod 100$, the last congruence reduces to $b \equiv 23 \bmod 100$, which implies that our answer is **23**.



7. The diagram is reproduced at right for handy reference, where we have labeled $m\angle MFO = \theta$. Since $AD = 2$ we find that $MF = 1$, so the right angle at M implies $\tan\theta = r$, the radius of the small circle. Since $\overline{OF}$ bisects $\angle MFG$ it follows

that $m\angle MFG = 2\theta$, thus $m\angle FGC = 2\theta$ as well. This is useful, since we already know that $FC = 1 + r$ and $\tan(2\theta) = \frac{FC}{GC}$, which should in theory permit us to compute GC, and thus $area(FCG)$, in terms of r.

The details go as follows. We first find that

$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2r}{1-r^2}.$$

Substituting this and $FC = 1 + r$ into $\tan(2\theta) = \frac{FC}{GC}$ gives

$$GC = \frac{FC}{\tan(2\theta)} = \frac{(1+r)(1-r^2)}{2r} = \frac{(1-r)(1+r)^2}{2r}.$$

Finally, equating the areas of $ADFE$ and FCG gives

$$(AD)(DF) = \tfrac{1}{2}(FC)(GC) \quad \Longrightarrow \quad 2(1-r) = \frac{(1-r)(1+r)^3}{4r},$$

which reduces to $8r = (1+r)^3$, since $r \neq 1$. Conveniently, this equation leads to a cubic that factors,

$$(r+1)^3 - 8r = r^3 + 3r^2 - 5r + 1 = (r-1)(r^2 + 4r - 1) = 0.$$

Since $r < 1$ our solution is the positive root of the quadratic factor, which is $r = \boldsymbol{\sqrt{5} - 2}$ via the quadratic formula.

Problem Credits

		5. Dr.V
1. Josh Z.	3. Josh Z.	6. Dr.V
2. Dr.V	4. Isaac L.	7. Isaac L.

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⋆ Regional Level ⋆

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The Mandelbrot Competition
www.mandelbrot.org
info@mandelbrot.org