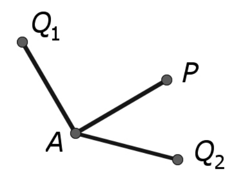
Team Play Topics

ROUND THREE

The first section of the Round Three Mandelbrot Team Play is reproduced below. A list of topics and practice problems are also provided to aid in preparation. Note that these problems are not meant to serve as a precise indicator of the problems that will appear on the contest. However, students who understand how to solve them should be able to make significantly more progress than they might have otherwise. So work hard on the problems, and good luck on the Team Play!

Facts: A *rotation* of the plane with center A through an angle θ leaves the center A fixed and carries any other point P to the point Q for which $QA = PA$ and $m\angle PAQ = \theta$. We call Q the *image* of P under the rotation. Here we adopt the usual convention that a positive angle corresponds to a counter-clockwise rotation, while a negative angle indicates a clockwise rotation. Thus in the diagram a $90°$ rotation about A carries P to Q_1, while a $-45°$ rotation carries P to Q_2.



TOPICS: rotations by $90°$, Cartesian plane, similar triangles, centroid

Practice Problems

1. Plot the point $P(5, 2)$ in the Cartesian plane. What are the coordinates of the image Q upon rotating P by $90°$ about the origin? Repeat this exercise when P has coordinates $(-7, 4)$ or $(3, -5)$. Then develop a general formula for the image of $P(x, y)$.

2. Show that the image upon rotating $P(x, y)$ by $90°$ about the point $A(a, b)$ has coordinates $Q(a+b-y, b-a+x)$. (TIP: shift both A and P until A coincides with the origin. Use the formula from the previous question, then shift back.)

3. There is a point P such that rotating P by $90°$ about $A(-3, -2)$, then rotating this image by $90°$ about $B(4, 1)$ returns P back to its original position. Find the coordinates of P.

4. Consider points $P_1(3, 1)$ and $P_2(2, 5)$. Find the coordinates of a point A such that rotating P_1 by $90°$ about A yields an image on the x-axis, while rotating P_2 by $90°$ about A yields an image on the y-axis. Illustrate your answer to visually check that it works.

5. Given triangle ABC, label the midpoints of $\overline{BC}$, $\overline{AC}$ and $\overline{AB}$ as L, M and N, respectively. Rotate B by $90°$ about L to obtain R, and rotate C by $90°$ about M to obtain S. Prove that if R is rotated by $90°$ about N the image is S.

Hints and answers on the next page. $\Longrightarrow$

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Hints and Answers

1. The images are $(-2,5)$, $(-4,-7)$ and $(5,3)$, in order. In general the image of $P(x,y)$ is $Q(-y,x)$.

2. Shifting both points by $-a$ in the x-direction and $-b$ in the y-direction changes the problem to rotating $(x-a,y-b)$ by $90°$ about the origin, which gives $(b-y,x-a)$. Now shifting back to the original position gives $(b+a-y,b-a+x)$, as claimed.

3. Rotating $P(x,y)$ as indicated yields an image of $(4-x,-8-y)$. If this is to match point P we must have $4-x=x$ and $-8-y=y$, giving the unique solution of $P(2,-4)$.

4. Using our formula leads to an answer of $A(4,1)$. Sketching all three points confirms that this point does satisfy the statement of the problem.

5. We omit the details of this proof. Carefully keeping track of all coordinates should reveal that rotating R by $90°$ about N gives precisely the coordinates of S. There are various labor-saving shortcuts available. For instance, one can set up coordinate axes so that A is the origin and B lies on the x-axis, which means that there are only three variables to handle instead of six. (It is also possible to find a purely geometric proof of this proposition without resorting to coordinates.)