Team Play Topics

Round Two

The first section of the Round Two Mandelbrot Team Play is reproduced below. A list of topics and practice problems are also provided to aid in preparation. Note that these problems are not meant to serve as a precise indicator of the problems that will appear on the contest. However, students who understand how to solve them should be able to make significantly more progress than they might have otherwise. So work hard on the problems, and good luck on the Team Play!

Facts: *Vieta's formulas* provide a means of computing symmetric expressions involving the roots of a polynomial. More precisely, the polynomial $c_0x^n + c_1x^{n-1} + c_2x^{n-2} + \cdots + c_{n-1}x + c_n$ has n roots. According to Vieta's formulas, the sum of these roots is $-c_1/c_0$, the sum of products of pairs of distinct roots is c_2/c_0, and the product of the roots is $(-1)^n c_n/c_0$.

Topics: Vieta's formulas, graphs of rational functions, inequalities

Practice Problems

1. Find the equations of all asymptotes for the graph of $y = \dfrac{3(x-1)}{(x+1)(x-2)} + \dfrac{2(x-1)}{(x-2)(x-4)}$.

2. In the remaining problems let a_1, a_2, ..., a_n be the n solutions to the equation

$$4(x-1)(x-2)\cdots(x-n) = 1,$$

listed in increasing order, where n is a positive integer. To begin, draw a reasonable graph of the left-hand side of this equation.

3. Explain why we expect all solutions of this equation to be real numbers.

4. Compute the value of this expression in terms of n.

$$\frac{1}{a_2a_3\cdots a_n} + \frac{1}{a_1a_3\cdots a_n} + \cdots + \frac{1}{a_1a_2\cdots a_{n-1}}$$

5. Suppose that n is even. What is the relationship between a_k and a_{n-k}?

6. Now let n be odd. Demonstrate that $a_1 > 1 + \dfrac{1}{4(n-1)!}$.

Hints and answers on the next page. $\Longrightarrow$

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Hints and Answers

1. This is a devious problem. Upon combining the two fractions (or better yet, splitting them up via partial fractions) one discovers that the factor of $(x-2)$ in the denominator cancels. Hence there are three asymptotes: two vertical at $x=-1$ and $x=4$, and one horizontal at $y=0$.

2. Your graph should hit the x-axis at $x=1, 2, \ldots, n$. It will go to positive infinity on both sides for n even, but will go to negative infinity on the left when n is odd. Finally, the heights of the "humps" will be larger near $x=1$ and $x=n$ and smaller near the middle.

3. If we can show that the height of every hump is at least 1, then the claim will follow. (Examine the graph and handle the cases of n even or odd separately.) Imagine plugging in $x=k+\frac{1}{2}$ for some positive integer k. Then at most two factors will equal $\frac{1}{2}$, while the rest will all be larger than 1. Multiplying by 4 ensures that the entire product is at least 1.

4. We rewrite the sum as $(a_1+a_2+\cdots+a_n)/(a_1a_2\cdots a_n)$. By Vieta's formulas we may determine the value of the numerator and denominator separately:

$$a_1+a_2+\cdots+a_n=-\frac{-4-8-\cdots-4n}{4}=\frac{n(n+1)}{2}, \qquad a_1a_2\cdots a_n=\frac{4n!-(-1)^n}{4}.$$

Therefore the desired expression is $\dfrac{2n(n+1)}{4n!-(-1)^n}$.

5. One should verify that if a is a solution to the equation, then so is $(n+1)-a$. One must then confirm that the solutions pair up in the appropriate manner; i.e. that $(n+1)-a_k$ really is a_{n-k}. We can then conclude that $a_k+a_{n-k}=n+1$.

6. For the sake of convenience, write $\epsilon=\frac{1}{4(n-1)!}$. Substituting $x=1+\epsilon$ into the left-hand side of the equation and negating all but one of the factors (an even number of them) yields

$$4(\epsilon)(1-\epsilon)(2-\epsilon)\cdots((n-1)-\epsilon).$$

This product is just slightly less than

$$4(\epsilon)(1)(2)\cdots(n-1)=4\epsilon(n-1)!=1.$$

In other words, at $x=1+\epsilon$ the graph of the function is positive but less than 1, meaning that we have not quite reached the first solution a_1. Therefore $a_1>1+\frac{1}{4(n-1)!}$.