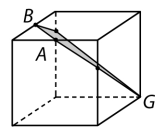
Answer Key	
1. 5	4. 19
2. \$26.50	5. 25/4*
3. $\frac{2}{5}$, $\frac{4}{9}$	6. $\frac{2}{5}$
	7. 512

*See below for an alternate acceptable answer.

1. The intersection of the plane with the cube is pictured at right. The polygon has sides along the top, front, left, right, and back sides. Therefore this polygon is a pentagon, having **5** sides.



2. Imagine that Marge only purchases the first type of bag; then she will need 7 bags in order to have enough apples and bananas. If she instead buys 6 bags then she must purchase 1 bag of the second type to have enough fruit. Continuing in this manner, we find the following possibilities, where each pair of numbers indicates the number of bags of the first and second type that must be bought.

$$(7,0) \quad (6,1) \quad (5,1) \quad (4,2) \quad (3,2) \quad (2,3) \quad (1,3) \quad (0,4)$$

Checking the price for each possibility reveals that the $(1,3)$ option is the cheapest, costing $1(\$4.00)+3(\$7.50)=\$\mathbf{26.50}$.

3. Applying the process to the fraction $\frac{m}{n}$ twice in a row gives

$$\frac{m}{n} \to \frac{n-m+1}{n+m+2} \to \frac{(n+m+2)-(n-m+1)+1}{(n+m+2)+(n-m+1)+2} = \frac{2m+2}{2n+5}.$$

To obtain the original fraction again we must have

$$\frac{m}{n} = \frac{2m+2}{2n+5} \implies 2mn+5m = 2mn+2n,$$

or $5m = 2n$. Since we want $\frac{m}{n}$ in lowest terms we take $m=2$, $n=5$, giving $\frac{2}{5}$. Applying our process repeatedly gives $\frac{2}{5} \to \frac{4}{9} \to \frac{2}{5} \to \frac{4}{9} \to \cdots$. Thus the two answers are $\mathbf{\frac{2}{5}}$ and $\mathbf{\frac{4}{9}}$.

4. Label the length and width of the rectangular pen as x and y. Then we are told that $x+2y=20$ and that $(\frac{1}{2}x)^2+y^2=9^2$. Rather than solve for x and y, we instead combine the given equations in a manner that gives the value of xy, which is the area we seek. Squaring the first equation and multiplying the second equation by 4 yields

$$x^2+4xy+4y^2=400, \qquad x^2+4y^2=324.$$

Subracting gives $4xy=76$, hence the area is $xy=\mathbf{19}$.

5. Let v represent the value of this infinite series. Then we have

$$\begin{aligned} v &= \alpha^2+2\alpha^5+3\alpha^8+4\alpha^{11}+\cdots \\ \alpha^3 v &= \phantom{\alpha^2+{}}\alpha^5+2\alpha^8+3\alpha^{11}+\cdots \\ \implies (1-\alpha^3)v &= \alpha^2+\alpha^5+\alpha^8+\alpha^{11}+\cdots, \end{aligned}$$

where we multiply the top equation by α^3 to obtain the second line, then subtract it from the first equation to reach the final line. We now have a geometric series with first term α^2 and common ratio α^3, hence

$$(1-\alpha^3)v = \frac{\alpha^2}{1-\alpha^3} \qquad \implies \qquad v = \frac{\alpha^2}{(1-\alpha^3)^2}.$$

(The reader should confirm that $0<\alpha<1$, ensuring that this geometric series actually converges.) But we know that $1-\alpha^3=\frac{2}{5}\alpha$, since α satisfies the equation $x^3+\frac{2}{5}x-1=0$. Hence

$$v = \frac{\alpha^2}{(\frac{2}{5}\alpha)^2} = \mathbf{\frac{25}{4}}.$$

The answers $6\frac{1}{4}$ or 6.25 are also acceptable answers.

6. Some algebraic and trignometric sleight of hand should do the trick. Applying the Law of Cosines to $\triangle ACD$ and $\triangle ABC$ shows that

$$\begin{aligned} 4^2+5^2-2(4)(5)\cos\varphi &= AC^2, \\ 2^2+3^2-2(2)(3)\cos\theta &= AC^2. \end{aligned}$$

Subtracting the second equality from the first and dividing through by 4 reveals that $10\cos\varphi - 3\cos\theta = 7$. On the other hand, subtracting the areas tells us that

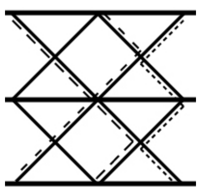
$$area(ACD) - area(ABC) = \frac{1}{2}(4)(5)\sin\varphi - \frac{1}{2}(2)(3)\sin\theta,$$

hence $10\sin\varphi - 3\sin\theta = 6$. Squaring these two relationships, adding the results, and simplifying leads to

$$100 - 60(\cos\theta\cos\varphi + \sin\theta\sin\varphi) + 9 = 85.$$

Rearranging gives our answer $\cos(\theta - \varphi) = \mathbf{\frac{2}{5}}$.

7. We claim that there are always eight ways to traverse each "level" of the braid, where the levels are separated by horizontal bars in the diagram at left. If the three paths enter a level at three distinct points (as occurs in the top level) then the middle path could branch in one of two ways to reach the middle of that level. There are then four ways to complete the level, since the two paths at the right crossing must each head in a different direction and the single path at the left crossing may head in either direction.



It is also possible for two of the paths to enter a level at the middle, while the third path enters at a side, as occurs for the bottom level. Almost identical reasoning shows that once again there are eight ways to traverse the level. Therefore a braid with n levels admits 8^n ways to draw paths from top to bottom. In our cases there are three levels, hence $8^3 = \mathbf{512}$ ways.

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⋆ National Level ⋆

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Round Five Solutions

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