Team Play Topics
Round One

A list of practice problems is provided below to aid in preparation for round one of the 2010 Mandelbrot Team Play. Note that these problems are not meant so serve as a precise indicator of the problems that will appear on the contest. However, students who understand how to solve them should be able to make significantly more progress than they might have otherwise. So work hard on the problems, and good luck on the Team Play!

Topics: Sets, partitions of sets, congruences applied to recurrences

Practice Problems

To *partition* the set $\{1, 2, 3, \ldots, n\}$ is to split it into a collection of one or more subsets, so that each number from 1 to n appears in exactly one of the subsets. For instance, a partition of $\{1, 2, 3, 4, 5, 6\}$ into three subsets might be $\{1, 3, 6\}\{2, 4\}\{5\}$. A *partition sum* is obtained by multiplying together the numbers in each subset of the partition, then adding together the results. Thus the partition sum for $\{1, 3, 6\}\{2, 4\}\{5\}$ is $(1)(3)(6) + (2)(4) + (5) = 31$.

1. What are the smallest and largest partition sums that may be obtained from $\{1, 2, 3, 4, 5\}$?
2. Figure out how to obtain a partition sum of $\{1, 2, 3, 4, 5, 6\}$ equal to 42.
3. Show that there exists a partition sum of $\{1, 2, 3, \ldots, 2n\}$ equal to $2(1^2 + 2^2 + \cdots + n^2)$.
4. (Open) Which positive integers can be obtained as a partition sum of a set $\{1, 2, 3, \ldots, n\}$?
5. Define a sequence of positive integers as $a_1 = 1$, $a_2 = 1$, and $a_{n+1} = a_n a_{n-1} + 1$ for all $n \geq 2$. Determine the first term of the sequence that is divisible by 10. (Tip: don't actually compute the terms, just figure out their values mod 10.)
6. Define a sequence of positive integers by setting $b_1 = 1$, $b_2 = 3$ and $b_{n+1} = 3b_n - b_{n-1}$ for $n \geq 2$. Find, with proof, all terms of the sequence that are divisible by 5. (Tip: find a recurring pattern in the sequence mod 5.)

Hints and answers on the next page. $\Longrightarrow$

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Hints and Answers

1. If you came up with 15 and 120 you were close. But we can obtain 14 with $\{1,2\}\{3\}\{4\}\{5\}$ and 121 with $\{1\}\{2,3,4,5\}$.

2. One possibility is to use $\{1\}\{3\}\{2,4\}\{5,6\}$ giving $(1)+(3)+(2)(4)+(5)(6)=42$.

3. Use the fact that $(1)(2n)+(2)(2n-1)+\cdots+(n)(n+1)=2(1^2+2^2+\cdots+n^2)$. (Verification of this identity is left as an exercise.)

4. Instinct suggests that all positive integers besides a few small exceptions may be expressed as a partition sum. Trial and error reveals that 4, 8 and 13 are among the exceptions. Any students making significant progress are invited to share their findings by sending written results to `info@mandelbrot.org`.

5. It is not until we reach a_{21} that we finally encounter a term divisible by 10.

6. The values of the sequence reduced mod 5 are shown below.

$$1,\ 3,\ 3,\ 1,\ 0,\ 4,\ 2,\ 2,\ 4,\ 0,\ 1,\ 3,\ 3,\ 1,\ 0,\ 4,\ 2,\ldots$$

The pattern repeats every ten terms, so every fifth term is divisible by 5. (I.e. b_5, b_{10}, b_{15}, $\ldots$)