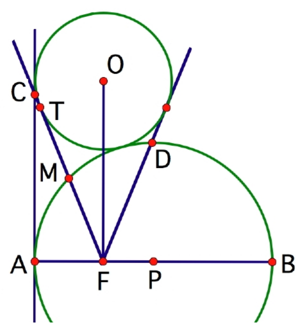
Team Play Solutions

Part i: Let U be the point of tangency between the two circles. Then clearly O, U and P line up, since radii $\overline{OU}$ and $\overline{UP}$ are perpendicular to the common tangent line at U. Hence $OP = r + R$. Next draw radius OV of the smaller circle from O across to the vertical line l. Observe that this creates rectangle $OVAF$; therefore $AF = r$ and $FP = R - r$. Finally, we may compute OF using the Pythagorean Theorem:

$$OF = \sqrt{(OP)^2 - (FP)^2} = \sqrt{(R+r)^2 - (R-r)^2} = \sqrt{4Rr} = 2\sqrt{Rr}.$$

Part ii: To begin, both $\triangle CAF$ and $\triangle FTO$ are right triangles, with right angles at $\angle A$ and $\angle T$, respectively. (The latter since the radius is perpendicular to the tangent at T.) Furthermore, angles $\angle ACF$ and $\angle TFO$ are both complementary to $\angle OFT$, and hence are congruent. It follows that the third pair of angles, namely $\angle AFC$ and $\angle TOF$, must also be congruent. Now recall that $OT = AF = r$, from which we can deduce that $\triangle CAF \cong \triangle FTO$ by the ASA congruence criterion.



Part iii: We use the fact that

$$\cos(m\angle CFP) = \cos(m\angle TFO + 90^\circ) = -\sin(m\angle TFO).$$

Since $\triangle TFO$ is a right triangle, $\sin m\angle TFO = OT/OF$. But $OT = r$ and $OF = 2\sqrt{rR}$ according to Part i. Combining these facts gives

$$\cos(m\angle CFP) = -\sin(m\angle TFO) = -\frac{r}{2\sqrt{rR}} = -\frac{1}{2}\sqrt{\frac{r}{R}},$$

as desired

Part iv: We will determine the value of PM using the Law of Cosines. First note that $CF = OF$ using the congruent triangles from Part ii. Hence $CF = 2\sqrt{rR}$ and thus $MF = \sqrt{rR}$ as M is the midpoint of $\overline{CF}$. We may now compute

$$\begin{aligned}
PM^2 &= (PF)^2 + (FM)^2 - 2(PF)(FM)\cos(m\angle MFP) \\
&= (R-r)^2 + (\sqrt{rR})^2 - 2(R-r)(\sqrt{rR})\left(-\frac{1}{2}\sqrt{\frac{r}{R}}\right) \\
&= R^2 - 2rR + r^2 + rR + (R-r)(r) \\
&= R^2,
\end{aligned}$$

giving $PM^2 = R^2$, so $PM = R$, the radius of the large circle. Therefore $PM = PA$ as claimed.

Part v: No doubt there are at least a half-dozen ways to establish this key step. Here is one relatively clean method. First we observe that $\angle OFD \cong \angle OFM$ due to the symmetry of the tangent lines. This implies that when we reflect point M over diameter $\overline{AB}$ the resulting point M', which is still on the circle, will line up with F and D. In other words, $m\angle ADF = m\angle ADM' = \frac{1}{2}m\widehat{AM'} = \frac{1}{2}m\widehat{AM}$. Next we take advantage of the fact that M is the midpoint of $\overline{CF}$, the hypotenuse of right triangle $\triangle ACF$, to deduce that $m\angle ACF = m\angle CAM = \frac{1}{2}m\widehat{AM}$. It follows that $\angle ADF \cong \angle ACF$.

Part vi: We are now able to reap the rewards of our hard work. In light of the congruent angles found in the previous part we immediately deduce that $AFDC$ is a cyclic quadrilateral. Hence opposite angles are supplementary, which implies that $m\angle CDF = 90^\circ$ since we know $m\angle CAF = 90^\circ$. This shows that $\triangle CDF$ is a right triangle.

An angle chase now ensues to show that $\triangle BDF$ is isosceles. The crux of the argument is to show that arcs $\widehat{AM}$ and $\widehat{MD}$ are congruent. We choose to prove this by showing that quadrilateral $PFMD$ is also cyclic. This follows from the fact that

$$m\angle MDF = \frac{1}{2}m\widehat{MM'} = m\widehat{AM} = m\angle MPF,$$

where point M' is defined above. Therefore

$$m\overparen{MD} = m\angle MPD = m\angle MFD = 2m\angle MFO = 2m\angle CAM = m\overparen{AM}.$$

We can now compare the angles of triangle BFD:

$$\begin{aligned} m\angle BFD &= \tfrac{1}{2}(m\overparen{AM}' + m\overparen{BD}) \\ &= \tfrac{1}{2}(m\overparen{AM} + m\overparen{BD}) \\ &= \tfrac{1}{2}(m\overparen{MD} + m\overparen{BD}) \\ &= \tfrac{1}{2}m\overparen{BM} \\ &= \tfrac{1}{2}m\overparen{BM}' \\ &= m\angle BDF. \end{aligned}$$

Therefore $\triangle BFD$ is indeed isosceles.

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Round Three Solutions

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