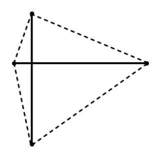
Answer Key	
1. False	4. $1 < x < 3$
2. 36	5. $4\frac{1}{4}$
3. 67	6. 24
	7. 17

1. We can easily obtain such a quadrilateral by drawing the diagonals first, for instance as shown by the solid segments at right. Connecting the endpoints with dotted segments then gives a quadrilateral with congruent, perpendicular diagonals. Evidently the answer is **False**.

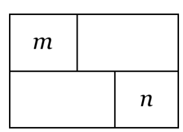


2. Recall that $0 + 1 + 2 + \cdots + 9 = 45$. Let's use this fact to check whether 41 is the answer. If 41 were the sum of the other eight digits, then $41+4+1$ would be the sum of all ten digits. So we need $41+4+1$ to equal 45, when in fact it gives 46. However, one quick approach is to work backwards from 45 trying out each two-digit number. Another method is to write our two-digit number as AB, so we would need

$$(10A + B) + A + B = 45 \qquad \Longrightarrow \qquad 11A + 2B = 45.$$

The only solution is $A = 3$, $B = 6$, so our two-digit number is **36**.

3. Let's organize our count according to the difference between the diagonally opposite numbers, labeled as m and n in the diagram.



Case 1: If $n = m + 2$, such as the $m = 2$, $n = 4$ example given in the problem, then both remaining squares must be filled with the value $m+1$. This leads to 3 possibilities. Similarly, when $n = m - 2$ there are again 3 possibilities.
Case 2: If $n = m+1$, such as when $m = 1$, $n = 2$, then each remaining square must be labeled either m or $m + 1$, and all 4 possibilities work. There are four possible values for m, giving 16 more ways. Similarly, when $n = m - 1$ there are 16 ways.
Case 3: Finally, we might have $n = m$. There are lots of subcases to consider; the reader should find that when $m = 2$, 3, or 4 then there are 7 good labelings, while $m = 1$ or 5 leads to 4 good labelings.
All together we have found $3 + 3 + 16 + 16 + 3(7) + 2(4) =$ **67** ways.

4. We begin by searching for values of x for which $5^x + 3 = 2^{2x+1}$. Remarkably, we find two integer solutions; namely $x = 1$ and $x = 3$, as

$$5^1 + 3 = 8 = 2^3, \qquad \text{and} \qquad 5^3 + 3 = 128 = 2^7.$$

Trying out $x = 2$ we discover that $5^2 + 3 = 28 < 2^5$. Hence $x = 2$ is part of the solution set, which therefore must be $\mathbf{1 < x < 3}$.

5. There are a total of twelve shaded regions from left to right, separated by the vertical segments. Observe that turning the fifth region over and placing it on top of the twelfth region yields a perfect 1×1 square, obviously with area 1. The same can be done with the sixth and eleventh regions, the seventh and tenth regions, and the eighth and ninth regions, giving an area of 4 so far. Finally, the four remaining triangles have area

$$\tfrac{1}{2}(\tfrac{1}{5})(\tfrac{1}{12}) + \tfrac{1}{2}(\tfrac{2}{5})(\tfrac{2}{12}) + \tfrac{1}{2}(\tfrac{3}{5})(\tfrac{3}{12}) + \tfrac{1}{2}(\tfrac{4}{5})(\tfrac{4}{12})$$
$$= \tfrac{1}{120}(1 + 4 + 9 + 16) = \tfrac{30}{120} = \tfrac{1}{4},$$

for an overall total area of $\mathbf{4\frac{1}{4}}$ or **4.25**.

6. Suppose that a positive integer n is quirky, meaning that the quotient and remainder have the same value when dividing n into 1,000,000. Let's call this common value s, meaning that

$$1{,}000{,}000 = n \cdot s + s \qquad \Longrightarrow \qquad \frac{1{,}000{,}000}{s} = n + 1.$$

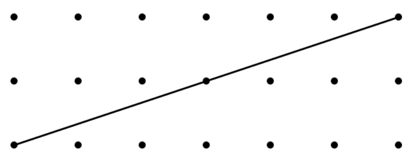
It appears that we obtain a quirky number for each divisor s of 1,000,000. Since $1{,}000{,}000 = 2^6 5^6$ has $(6+1)(6+1) = 49$ divisors, this would seem to give 49 quirky numbers. However, the remainder s must be less than the divisor n, which rules out around half of the options, including $s = 1000$, $n = 999$. Hence there are actually only **24** quirky numbers.

7. The persistent solver might be able to discover a quadrilateral that precisely fits the description given in the problem. (If so, let me know; I've forgotten how it's done.) Fortunately, we can determine the area without knowing exactly how the quadrilateral looks by using Pick's Theorem. Recall that this formula states

$$\mathit{area} = I + \tfrac{1}{2}B - 1,$$

where I is the number of interior points and B is the number boundary points of the quadrilateral, including the vertices.

We are given that there are eight interior points, so $I = 8$. The key to finishing is to notice that a lattice segment of length $2\sqrt{10}$ can only arise if it has slope $\frac{1}{3}$ (as shown below), or slope $-\frac{1}{3}$, or slope ± 3.



Regardless, such a segment involves three boundary points. In a similar fashion a segment of length $3\sqrt{13}$ has four boundary points, a segment of length $4\sqrt{2}$ has five boundary points, and a segment of length 11 has twelve boundary points. But each vertex is counted twice along the way, hence

$$B = 3 + 4 + 5 + 12 - 4 = 20,$$

giving an area of $8 + \frac{1}{2}(20) - 1 = \mathbf{17}$.

Problem Credits

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