Team Play Topics

Round Three

The first section of the Round Three Mandelbrot Team Play is reproduced below. A list of topics and practice problems are also provided to aid in preparation. Note that these problems are not meant to serve as a precise indicator of the problems that will appear on the contest. However, students who understand how to solve them should be able to make significantly more progress than they might have otherwise. So work hard on the problems, and good luck on the Team Play!

Facts: Given three non-collinear points A, B and C in the plane, there are precisely four circles tangent to all three of the lines AB, AC and BC. The one within $\triangle ABC$ is the *incircle*, while the remaining three are called *excircles*. The centers and radii of these circles are commonly denoted I, I_a, I_b, I_c and r, r_a, r_b, r_c. There are many relationships among the lengths defined by the points of tangency. Thus in the diagram below we have $BT_a = CU_a$.

Topics: incircle/excircles, circumcircle, similar triangles, equally spaced lines, circle facts

Practice Problems

1. On a sheet of blank paper plot three points A, B and C towards the middle forming a "nothing" triangle. (Not equilateral, not isosceles, not right, not obtuse.) Next draw lines AB, AC and BC. At point A draw the two lines that bisect all four angles, then do the same at points B and C. What sort of figure do these six angle bisector lines form? Explain why this occurs.

2. Draw the circle through points A, B and C, called the *circumcircle*, and let M be the point where the angle bisector of $\angle BAC$ intersects this circle. Show that M is the midpoint of arc $\widehat{BC}$.

3. Next consider the point N where the other (exterior) angle bisector at A meets the circumcircle. Prove that N is the midpoint of major arc $\widehat{BC}$; i.e. the arc that goes the "long way around the circle" from B to C.

4. Let L be the midpoint of $\overline{AN}$, let D be the point diametrically opposite A on the circumcircle, and call the center of this circle O. Explain why lines AM, LO and ND are equally spaced lines. What does this imply about the segments these lines cut off on side $\overline{BC}$?

5. The various angle bisectors drawn in the first problem above are concurrent at the incenter I and the excenters I_a, I_b and I_c. Label these points in your diagram (I is inside $\triangle ABC$, while I_a is on the other side of $\overline{BC}$ from A, and so on), then sketch the circles with these centers that are tangent to all three of lines AB, AC and BC.

6. Let T_a be the point of tangency of the incircle with side $\overline{BC}$, and let U_a be the point of tangency of the excircle around I_a with side $\overline{BC}$. Figure out why $BT_a = CU_a$.

Hints and answers on the next page. $\Longrightarrow$

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Hints and Answers

1. The six angle bisector lines form a larger triangle along with its three altitudes. The perpendicularity results from the fact that the adjacent angles at A sum to $180°$ (before we drew the angle bisectors), so half these angles sum to $90°$.

2. Note that angles $\angle BAM$ and $\angle MAC$ are congruent. Since these angles intercept arcs $\overparen{BM}$ and $\overparen{MC}$, the arcs are also congruent, which makes M the midpoint.

3. There are several ways to prove this using various inscribed angles. An alternate approach is to argue that since the two angle bisectors at A are perpendicular we have $m\angle MAN = 90°$, so $\overparen{MN}$ has measure $180°$, meaning that M and N are diametrically opposite one another on the circle. Since M is the midpoint of arc $\overparen{BC}$, it follows that N is the midpoint of major arc $\overparen{BC}$. (The reader should fill in the details.)

4. Here is a sketch of the solution. First argue that all three lines are perpendicular to line AN, which will show that they are parallel. We have already seen that line AM is, and we get a right angle at N since $\overline{AD}$ is a diameter of the circle. Next show triangles $\triangle OLA$ and $\triangle OLN$ are congruent to deduce that we have a right angle at L. Finally, since L is the midpoint of $\overline{AN}$, these parallel lines are equally spaced, which means that the segments they cut off on line BC (or any other transversal, for that matter) will be congruent.

5. Sketch left to the reader.

6. This is a standard fact regarding incircles and excircles. We refer the reader to an online source such as `http://www.cut-the-knot.org/triangle/InExCircles.shtml` for an explanation.