Team Play Solutions

Part i: We compute the quotients and discover that in each case the quantity $a^2+b^2+c^2$ is exactly twice $abc+1$.

$$\frac{1^2+1^2+2^2}{1\cdot 1\cdot 2+1}=\frac{6}{3}=2, \qquad \frac{1^2+2^2+3^2}{1\cdot 2\cdot 3+1}=\frac{14}{7}=2,$$

$$\frac{1^2+3^2+4^2}{1\cdot 3\cdot 4+1}=\frac{26}{13}=2, \qquad \frac{1^2+4^2+5^2}{1\cdot 4\cdot 5+1}=\frac{42}{21}=2.$$

Since $q=2=1^2+1^2$, the q-values are indeed the sum of two squares.

Part ii: The previous part suggests that $(1,t,t+1)$ is an excellent triple for any positive integer t. We can confirm this by computing

$$\frac{1^2+t^2+(t+1)^2}{(1)(t)(t+1)+1}=\frac{2t^2+2t+2}{t^2+t+1}=2.$$

As expected, the quotient is a positive integer, showing that $(1,t,t+1)$ is an excellent triple.

Part iii: It seems that triples of the form $(2,t,g(t))$ are excellent, for a certain function $g(t)$. To find a formula for $g(t)$ we use finite differences.

$t=1$		2		3		4		5
10		32		78		160		290
	22		46		82		130	
		24		36		48		
			12		12			

Extending the pattern back one term in each row gives values of 0, 10, 12, 12 along the diagonal beneath $t=0$, leading to the formula

$$\begin{aligned} g(t) &= 0\binom{t}{0}+10\binom{t}{1}+12\binom{t}{2}+12\binom{t}{3} \\ &= 10t+12\cdot\frac{t(t-1)}{2}+12\cdot\frac{t(t-1)(t-2)}{6} \\ &= 10t+6t^2-6t+2t^3-6t^2+4t \\ &= 2t^3+8t. \end{aligned}$$

We can confirm that triples of the form $(2,t,2t^3+8t)$ are in fact excellent by simplifying

$$\frac{2^2+t^2+(2t^3+8t)^2}{2t(2t^3+8t)+1}=\frac{4t^6+32t^4+65t^2+4}{4t^4+16t^2+1}=t^2+4.$$

Notice that the q-value is t^2+2^2, which is a sum of two squares.

Part iv: By working with q we can actually avoid messy algebra. We are given that $q=\frac{a^2+b^2+c^2}{abc+1}$, thus $a^2+b^2+c^2=qabc+q$. Hence we have

$$\begin{aligned} a^2+b^2+(qab-c)^2 &= a^2+b^2+c^2-2qabc+q^2a^2b^2 \\ &= qabc+q-2qabc+q^2a^2b^2 \\ &= q(qa^2b^2-abc+1). \end{aligned}$$

Notice that this expression is exactly q times $ab(qab-c)+1$. Hence the former expression is divisible by the latter, as desired. Even more importantly, their quotient is q. In other words, the q-value for the triple $(a,b,qab-c)$ is the same as the q-value for (a,b,c). This will play an important role shortly.

Part v: Consider the function $\frac{a^2+b^2+x^2}{abx+1}$, for fixed positive integers a,b. Clearly plugging in $x\geq 0$ will yield a positive value, while $x\leq -1$ gives a negative value. (Or an undefined expression, in one instance.) But we have just seen in the previous part that substituting $x=qab-c$ in this expression yields q, a positive value, so we deduce that $qab-c\geq 0$.

Recall that we have an excellent triple (a,b,c) with quotient q. For this value of q the equation

$$\frac{a^2+b^2+x^2}{abx+1}=q,$$

leads to a quadratic in x, which has two solutions; namely, $x=c$ and $x=qab-c$. If we can show that the smaller root of this quadratic is less than b then we will be done, since we are assuming that $c\geq b$. According to the quadratic formula the actual root is

$$\tfrac{1}{2}\left(qab-\sqrt{q^2a^2b^2-4a^2-4b^2+4q}\right).$$

One then checks that the following inequalities are equivalent:

$$\begin{aligned}
\tfrac{1}{2}\left(qab - \sqrt{q^2a^2b^2 - 4a^2 - 4b^2 + 4q}\right) &< b, \\
qab - 2b &< \sqrt{q^2a^2b^2 - 4a^2 - 4b^2 + 4q}, \\
q^2a^2b^2 - 4ab^2q + 4b^2 &< q^2a^2b^2 - 4a^2 - 4b^2 + 4q, \\
a^2 - q &< b^2(aq - 2).
\end{aligned}$$

Now if $a = 1$ the last line is clearly true, as $q \geq 2$. And if $a > 1$ then the left-hand side is less than a^2, while the right-hand side is at least b^2. Since $a \leq b$ we conclude that the final inequality is true either way, hence so is the initial one, and we're done.

Part vi: The groundwork has now been laid for a beautiful finish to the problem. Suppose we have an excellent triple of numbers with $q \geq 2$, and list them in increasing order as (a, b, c). Now compute $qab - c$. We have shown that this quantity is either positive or 0. If it is positive, then we have a new excellent triple $(qab - c, a, b)$ with the same q-value, but the sum of the elements has decreased. We can't repeat this process forever, so eventually we must encounter the other possibility, namely that $qab - c = 0$, giving the triple $(0, a, b)$. But note that the quotient is still q, which means that

$$\frac{0^2 + a^2 + b^2}{(0)(a)(b) + 1} = q \qquad \Longrightarrow \qquad q = a^2 + b^2.$$

Therefore the original quotient can be written as $q = a^2 + b^2$, as claimed.

N.B. Of course we still must handle the case $q = 1$ from part v. Trouble only arises if $a = 1$ or $a = 2$. We leave it to the reader to check that there are no values of b and c for which $\frac{a^2+b^2+c^2}{abc+1} = 1$ in these cases, so the argument in part vi actually always works, and you're done.

By the way, this result generalizes problem six from the 1988 IMO, one of the most legendary (and difficult) IMO problems ever posed. So congratulations if you made significant progress on this Team Play!

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Round One Solutions

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