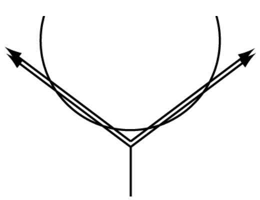
Answer Key

1. 25	4. 38
2. 9	5. 6
3. 8	6. 10089
	7. $2\sqrt{2}$

1. In five hours Warren can build 20 puzzles, which requires exactly two boards worth of wood. Hence he spends \$26 on materials. On the other hand, the 20 puzzles net him 20(\$7.50) = \$150 in income. His profit is therefore \$150 − \$26 = \$124 in total, which comes to $\frac{1}{5}$(\$124) ≈ \$**25** per hour, to the nearest dollar.

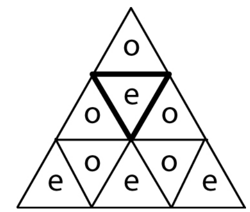
2. We are interested in letters such as R, since R does not appear in the word INFINITESIMAL, but doesn't not appear in both of them. (Because it does appear in GARGANTUAN.) A letter such as V is no good, because V does not appear in the first word but it also does not appear in the second word. A moment's thought reveals that the question is actually just asking for those letters in one word or the other but not in both, albeit in a somewhat disguised manner. Those letters are R,U,G,S,M,I,L,E,F, for a total of **9** letters.

3. Any circle that encloses the common endpoints of the rays (so as to intersect all five rays at least once) can do no better than to create five points of intersection. However, by ignoring the lower ray and intersecting each of the others twice, as shown in the diagram at left, it is possible to obtain as many as **8** points of intersection between a circle and this object.



4. It is possible to rule out many values of n by observing that if either n or $n+1$ were a prime then the sum of the primes dividing these two numbers would be at least n, which is too large. So the first few pairs we need consider are $(8,9)$, $(9,10)$, $(14,15)$ and $(15,16)$. The sums of the primes dividing them are 5, 10, 17, 10 respectively, none of which equals $n-1$. But from this point on the sum will be too small unless n and $n+1$ are of the form $2p$ and $3q$ for primes p and q. So we proceed to check $(21,22)$, $(33,34)$, $(38,39)$ and $(57,58)$. The sums in these cases are 23, 33, 37, 53 respectively. Apparently $n = \mathbf{38}$ satisfies the statement. (The sum is too small for all $n > 38$, which is why the answer is unique.)

5. Since there are many more multiples of 2 than multiples of 7 available to us, it makes sense to begin by trying to arrange the five odd digits and the four even digits so as to create as many even three-in-a-row sums as possible. There is no way to arrange for all twelve such sums to be even (why not?), but we can come pretty close using the arrangement shown at right, in which 'e' represents an even digit and 'o' represents an odd digit. The only odd sums here are the lower left and lower right horizontal three-in-a-row sums, which therefore must be odd multiples of 7. Such sums can be obtained using $4+2+1$, $8+7+6$, $9+7+5$ and $9+8+4$. We must choose two sums which have a digit in common, so our only option is to use the digits 8, 9, 4, 1, 2 in that order (or the reverse order) across the bottom. Either way, the only remaining even digit is **6**, which will appear in the highlighted triangle.



6. We claim that for all $n \geq 2$ we have $T_{n+1} = 2T_n + T_{n-1}$. This recursion immediately leads to the answer, for if $T_7 = 717$ and $T_8 = 1731$ then we may compute $T_9 = 2T_8 + T_7 = 2(1731) + 717 = 4179$, followed by $T_{10} = 2T_9 + T_8 = 2(4179) + 1731 = \mathbf{10089}$. It remains to demonstrate that the recursion is valid.

Let A_n be the number of valid colorings of a row of n squares in which the last two squares have different colors, and let B_n be the number of valid colorings ending with two squares of the same color. For starters, we have $T_n = A_n + B_n$. But we also deduce that

$$A_{n+1} = A_n + 2B_n \quad \text{and} \quad B_{n+1} = A_n + B_n.$$

For instance, imagine that a row of n colored squares ends with two different colors, say red-blue. To obtain a valid coloring of a row of $n+1$ squares we may either append a red square or a blue square, which contributes to either A_{n+1} or B_{n+1}, respectively. This explains the presence of the A_n term in each recursion above. (The reader is invited to supply the remaining details.) The second recursion shows $B_{n+1} = T_n$, and adding the recursions gives

$$A_{n+1} + B_{n+1} = 2(A_n + B_n) + B_n \qquad \Longrightarrow \qquad T_{n+1} = 2T_n + T_{n-1}.$$

7. Let α, β and γ be the measures of $\angle A$, $\angle B$ and $\angle C$ in $\triangle ABC$. First note that two of the angles of $\triangle CIA$ are $\frac{1}{2}\alpha$ and $\frac{1}{2}\gamma$, since the angle bisectors meet at I. Hence $m\angle CIA = 180° - \frac{1}{2}\alpha - \frac{1}{2}\gamma = 90° + \frac{1}{2}\beta$, since $\alpha + \beta + \gamma = 180°$. According to the Extended Law of Sines we have

$$\frac{AC}{\sin(m\angle CIA)} = 2R \qquad \Longrightarrow \qquad R = \frac{5}{2\sin(90° + \frac{1}{2}\beta)},$$

where R is the radius of circle CIA which we wish to compute. But $\sin(90° + \frac{1}{2}\beta) = \cos(\frac{1}{2}\beta)$, which we can compute once we know $\cos\beta$. By the Law of Cosines,

$$5^2 = 4^2 + 6^2 - 2(4)(6)\cos\beta \qquad \Longrightarrow \qquad \cos\beta = \frac{9}{16}.$$

Using the double angle formula for cosine we have $\cos\beta = 2\cos^2(\frac{1}{2}\beta) - 1$, so $\cos^2(\frac{1}{2}\beta) = \frac{1}{2}\left(\frac{9}{16} + 1\right) = \frac{25}{32}$, thus $\cos(\frac{1}{2}\beta) = \frac{5}{4\sqrt{2}}$ since we know that $\cos(\frac{1}{2}\beta)$ is positive. Putting everything together gives

$$R = \frac{5}{2(\cos(\frac{1}{2}\beta))} = \frac{5}{2\left(\frac{5}{4\sqrt{2}}\right)} = \mathbf{2\sqrt{2}}.$$

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