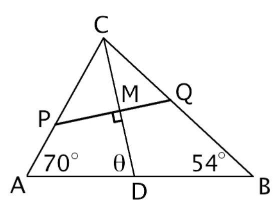
## Answer Key

| | |
|---|---|
| 1. 8 | 4. 81 |
| 2. 2125 | 5. 3, 5 |
| 3. 74° | 6. 61.2 |
| | 7. $2\frac{1}{2}$ |

1. Suppose the second statement were true. Since the only pair of digits that multiply to 28 are 4 and 7, these numbers would have to appear on the reverse sides of the cards. But their sum is 11, meaning both statements would be true, contradicting the fact that there is a 4 on one of the cards, making that card's statement false. Therefore the second statement must be false, so a 4 appears on that card. Now if the first statement were true then we would once again have a 4 and a 7. Hence both statements must be false, implying that both cards feature a 4, so the desired sum is **8**.

2. We need only determine how the new total area compares to the old. Suppose that each outlet originally had radius $R$, making the combined area equal to $2\pi R^2$. The new radii would be $2R$ and $\frac{1}{2}R$ after changing the sizes of the outlets, for a combined area of $\pi(2R)^2 + \pi(\frac{1}{2}R)^2$, or $4.25\pi R^2$. Since the area has increased by a factor of 2.125 so will the flow rate, giving $2.125(1000) =$ **2125** gpm.

3. Imagine unfolding the paper again after folding point $C$ on top of point $D$. By the nature of folds, the crease $\overline{XY}$ will be perpendicular to $\overline{CD}$. (The crease will bisect $\overline{CD}$ also, but we won't need this fact.) Since there are a total of 360° in quadrilateral $APMD$ we know that $m\angle APM = 200° - \theta$. But if quadrilateral $APQB$ is to be cyclic, then the sum of its opposite angles must be 180°. Hence $(200° - \theta) + 54° = 180°$, which leads to $\theta =$ **74°**.




4. We first combine each pair of numbers having the same tens digit, like (53)(57). One quickly discovers that each such product ends with the digits 21. This occurs because

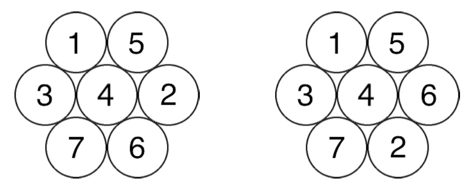
$$(10n + 3)(10n + 7) = 100n^2 + 70n + 30n + 21 = 100(n^2 + n) + 21.$$

Therefore to determine the final two digits of the entire product we must find the last two digits of $21^9$. It is relatively easy to find a pattern among the successive powers 21, $21^2$, $21^3$, etc. Alternately, one may write

$$(1 + 20)^9 = 1 + 9(20) + 36(20)^2 + \cdots + 20^9$$

using the Binomial Theorem. Only the first two terms contribute to the units and tens digits, and their sum is 181, so the answer is **81**.

5. It is surprisingly tricky to stumble upon even a single configuration of the seven digits satisfying the given condition. A careful search reveals a total of two possible solutions, not counting rotations and reflections. These are shown below.




Note that in either case the same two digits flank the 1, so our answer is **3,5**. (It is also fine to list the digits in the other order, as in **5,3**.)

6. Label the lengths of the sides of $\triangle ABC$ as $a$, $b$ and $c$ as usual. Call $AP = d$ and let $R$ be the circumradius of $\triangle ABC$. We claim that the area of quadrilateral $ABPC$ is given by $aR\sin^2(m\angle ABP)$. We briefly outline one means of establishing this result. To begin,

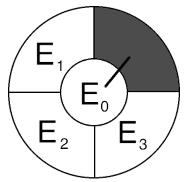
$$\textit{area}(ABPC) = \tfrac{1}{2}bd\sin(\tfrac{1}{2}\alpha) + \tfrac{1}{2}cd\sin(\tfrac{1}{2}\alpha) = \tfrac{1}{2}(b+c)d\sin(\tfrac{1}{2}\alpha),$$

where $\alpha = m\angle BAC$. Next we use Ptolemy's Theorem to deduce that $b + c = ad/e$, where $e = BP = CP$. The Extended Law of Sines gives

$e = 2R\sin(\frac{1}{2}\alpha)$ and $d = 2R\sin(m\angle ABP)$. Combining all these results gives the desired formula. It follows that the area is

$$\mathit{area}(ABPC) = aR\sin^2(m\angle ABP) = (17)(10)(\tfrac{3}{5})^2 = \mathbf{61.2}.$$

7. Let $E_0$ represent the expected number of times to pass or land on the shaded region in the course of the game if the marker is currently on the shaded region. (This is the quantity we wish to determine.) In the same way, let $E_1$, $E_2$, and $E_3$ measure the same quantity when the marker is situated at one of the remaining three spaces on the game board, as indicated in the diagram.




We now find that

$$E_1 = \tfrac{1}{4}(E_2) + \tfrac{1}{4}(E_3) + \tfrac{1}{4}(1) + \tfrac{1}{4}(E_1 + 1).$$

This equality holds because if the marker is currently on space $E_1$, then with probability $\frac{1}{4}$ we will advance one space. We haven't landed on or passed the shaded region, so the expected number of times to do so in the rest of the game is just $E_2$. The same reasoning applies if we advance two spaces. We advance three spaces with probability $\frac{1}{4}$; in this case the game is over and we hit the shaded square once. By rolling a 4 we return to square $E_1$ but have passed the shaded square, so the expected value in this case is $E_1 + 1$.

In the same manner we obtain the equalities

$$\begin{aligned}
E_0 &= \tfrac{1}{4}(E_1) + \tfrac{1}{4}(E_2) + \tfrac{1}{4}(E_3) + \tfrac{1}{4}(1), \\
E_2 &= \tfrac{1}{4}(E_3) + \tfrac{1}{4}(1) + \tfrac{1}{4}(E_1 + 1) + \tfrac{1}{4}(E_2 + 1), \\
E_3 &= \tfrac{1}{4}(1) + \tfrac{1}{4}(E_1 + 1) + \tfrac{1}{4}(E_2 + 1) + \tfrac{1}{4}(E_3 + 1).
\end{aligned}$$

It is routine to solve this system of equations, revealing that $E_0 = \mathbf{2\tfrac{1}{2}}$.



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⋆ National Level ⋆

# The Mandelbrot Competition

## Round Three Solutions

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