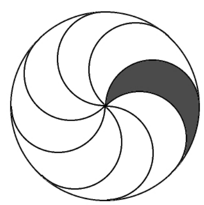
ANSWER KEY		4.	$\log_2 21$
1.	PEA	5.	$\sqrt{34}$
2.	2/5	6.	1238
3.	20π	7.	7070

1. Of the four letters mentioned—A, E, P, R—the letter that is *not* in Brenna's word must be part of each guess, since exactly one letter in each guess is wrong. Hence this letter is R, so Brenna's word contains the letters A, E, P. Only one ordering of the letters A, E, P results in at least one letter in the right position each time; namely, **PEA**.

2. There are eight card faces that Nick could have seen (two per card), three of which are black and five of which are red. We are only interested in the red faces. Of these five, two belong to the card that is red on both sides. Therefore the probability that Nick has chosen this card is $\mathbf{2/5}$.



3. The shaded region is bounded by three circular arcs. The two arcs within the large circle are semicircles with diameter 16mm, so each of these has length $\frac{1}{2}(16\pi)$ mm, which combine for 16π mm. The remaining portion of the boundary consists of an eighth of the circumference of the large circle, so it has length $\frac{1}{8}(32\pi) = 4\pi$ mm. Hence the total perimeter is $16\pi + 4\pi = \mathbf{20\pi}$ mm.

4. The given expression has the form $AB + A + B$, which reminds us of the product $(A+1)(B+1) = AB + A + B + 1$. Therefore we write

$$\begin{aligned}
&(\log_2 3)(\log_6 7) + (\log_2 3) + (\log_6 7) \\
&\quad = (\log_2 3 + 1)(\log_6 7 + 1) - 1 \\
&\quad = (\log_2 3 + \log_2 2)(\log_6 7 + \log_6 6) - 1 \\
&\quad = (\log_2 6)(\log_6 42) - 1 \\
&\quad = (\log_2 42) - \log_2 2 \;=\; \log_\mathbf{2} \mathbf{21}.
\end{aligned}$$

5. The underlying principle behind this problem states that given several similar shapes, the area of each shape is proportional to the square of a particular length within that figure. For instance, in the given diagram, the semiperimeter s of $\triangle ADC$ is some multiple λ of its inradius 5. (The exact value of λ is not relevant.) Therefore

$$area(ADC) = 5s = 5(5\lambda) = 25\lambda.$$

But triangles $\triangle ADC$ and $\triangle CDB$ are similar with constant of proportionality $\frac{3}{5}$. Hence the semiperimeter of $\triangle CDB$ is $\frac{3}{5}s = 3\lambda$, so its area is 9λ. In the same manner, since $\triangle ABC$ is similar to both of the other triangles, its area will be λr^2, where r is the inradius. But the sum of the smaller areas equals the larger area, giving

$$25\lambda + 9\lambda = \lambda r^2 \qquad \Longrightarrow \qquad r = \sqrt{\mathbf{34}}.$$

6. Clearly $N = 1$ and $N = 3$ represent winning positions, while $N = 2$ and $N = 4$ are losing positions. One can recursively extend these lists using the fact that a winning position is one in which it is possible to make a move that results in a losing position for the other player, while a losing position is one in which every move leaves a winning position for the other player. We find that

Win	$N = 1$	3	5	7	9	10	11	12	14	16	18	20 ...
Lose	$N = 2$	4	6	8	13	15	17	19	26	28	30	32 ...

It now becomes apparent that these sequences repeat mod 13 and that eight of each thirteen integers are winning positions. As $2002 = 13 \cdot 154$ we know that $8 \cdot 154 = 1232$ values in the range $1 \le N \le 2002$ are winning positions. Of the integers from 2003 to 2012 another six values are good, for a grand total of $1232 + 6 = \mathbf{1238}$ winning positions.

7. The first condition may be rewritten as

$$1 \le \frac{a+100}{b} < 2 \qquad \Longrightarrow \qquad b \le a + 100 < 2b,$$

while the second condition becomes $2a \le b + 100 < 3a$. We may visualize the pairs (a, b) of values satisfying these inequalities as lattice points in the plane within or on the upper/right edges of the region bounded by the four lines $y = x+100$, $y = \frac{1}{2}(x+100)$, $y = 2x-100$, and $y = 3x-100$. It is straight-forward to determine that the vertices of this quadrilateral are $(60, 80)$, $(100, 100)$, $(200, 300)$, and $(100, 200)$ in order around the perimeter. Standard techniques also show that the area of this region is 7000. Furthermore, there are 260 points (a, b) with integer coordinates along the boundary of this region. The reader should confirm that of these, 61 are situated along the left/lower edges, including the vertices $(100, 100)$ and $(100, 200)$, and thus do not satisfy the above inequalities. The remaining 199 are located along the upper/right edges and do satisfy the inequalities. Finally, by Pick's Formula we have

$$7000 = I + \tfrac{1}{2}(260) - 1 \qquad \Longrightarrow \qquad I = 6871,$$

where I counts the number of lattice points in the interior of the region. This gives a total of $6871 + 199 =$ **7070** pairs (a, b) that satisfy the conditions of the problem.

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⋆ National Level ⋆

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