January 2010

The Mandelbrot Team Play

Round One Test

Time Limit: 60 minutes

Setup: For a positive integer n we define B_n, the n^{th} *Bell number*, as the number of ways to partition the set $\{1, 2, 3, \ldots, n\}$ into a collection of one or more subsets. For instance, $B_3 = 5$ since there are five ways to split $\{1, 2, 3\}$ into subsets. These five partitions are

$$\{1\}\{2\}\{3\} \qquad \{1,2\}\{3\} \qquad \{1,3\}\{2\} \qquad \{2,3\}\{1\} \qquad \{1,2,3\}.$$

Similarly $B_2 = 2$ since there are two ways to partition $\{1, 2\}$; namely, as $\{1\}\{2\}$ and $\{1, 2\}$. Furthermore, $B_1 = 1$ since the only partition of $\{1\}$ is $\{1\}$. A number appearing in a subset all by itself, such as the $\{3\}$ in the first two partitions above, is called a *singleton*.

There are many delightful relationships among the Bell numbers. The following questions develop several of these, with the goal of predicting which Bell numbers are divisible by 3.

Problems

Part i: (4 points) Write out all the partitions of the set $\{1, 2, 3, 4\}$. Count the number of partitions that appear in your list to determine the value of B_4.

Part ii: (4 points) List all partitions of $\{1, 2, 3, 4\}$ that remain unchanged when the 3 and 4 trade places. For instance, $\{2\}\{1, 3, 4\}$ becomes $\{2\}\{1, 4, 3\}$, which is the same partition, since the order of the numbers within a subset doesn't matter. Next list all partitions of $\{1, 2, 3, 4, 5\}$ that remain unchanged when the 4 and 5 trade places.

Part iii: (5 points) Prove that there are B_n partitions of $\{1, 2, 3, \ldots, n+2\}$ in which both of the subsets $\{n+1\}$ and $\{n+2\}$ appear as singletons. Then explain why $n+1$ and $n+2$ appear together in the same subset for B_{n+1} of these partitions.

Part iv: (5 points) Swapping the numbers $n+1$ and $n+2$ essentially pairs up all the partitions of $\{1, 2, 3, \ldots, n+2\}$, except those partitions that are unaffected by the swap. Use this idea to prove that $B_{n+2} - B_{n+1} - B_n$ is even. Then predict which Bell numbers are even.

Part v: (5 points) Find an expression for the number of partitions of $\{1, 2, 3, \ldots, n+3\}$ that remain unchanged upon replacing $n+2$ by $n+1$, replacing $n+3$ by $n+2$, and replacing $n+1$ by $n+3$. Prove that your expression is correct.

Part vi: (5 points) Determine a combination of Bell numbers that is always divisible by 3, then use your relationship to determine, with proof, which Bell numbers are multiples of 3.