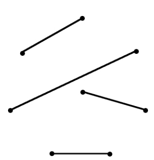
Answer Key	
1. 8	4. 17/101
2. 21	5. 3200
3. 21/25	6. $\frac{1}{2}(\sqrt{3}+1)$
	7. 42063

1. One productive way to find a route is to work backwards from the ending position. We can't have moved downwards as our final move, since 25 is not a multiple of 3, so we must have moved to the right with a score of 23. By the same reasoning the move before that must also have been to the right, and so forth. One route involving eight moves is right, left, down, right, left, down, right, right with scores

$$1 \to 3 \to 2 \to 6 \to 8 \to 7 \to 21 \to 23 \to 25.$$

No route uses fewer moves, so **8** moves is the answer.

2. We first note that the center point can be joined to any of the vertices of the heptagon, giving 7 ways to begin drawing the segments. For each such segment there are precisely three ways to join the remaining six vertices. One way is shown at left, the second is the mirror image over the segment extending from the center, and the third way involves joining adjacent vertices in pairs around the perimeter. Hence there are a total of $7 \cdot 3 = \mathbf{21}$ ways.



3. A quick sketch reveals that the issue of whether the two circles intersect depends completely upon the distance between their centers. If the point used as the center of the circle with radius 3 is at least 2 units away from the center of the circle with radius 5, then the circles will intersect. Such points comprise the region within the circle of radius 5 but outside a circle of radius 2 having the same center. Hence the area of this region is $25\pi - 4\pi = 21\pi$, leading to a probability of $21\pi/25\pi = \mathbf{21/25}$.

4. We know that the decimal expansion for $\frac{3}{17}$ will begin repeating after at most 16 digits. A careful inspection of the given decimal shows that we do need the full 16 digits. We are now instructed to take digits in places 1, 3, 9, 27, 81, 243, ... to create a new decimal. Since every 16$^{\text{th}}$ digit is the same, this amounts to taking digits 1, 3, 9, 11, 1, 3, ... from the decimal expansion. But as has now become evident, the powers of 3 repeat with period four. Therefore the desired decimal expansion is

$$.168316831683\ldots = \frac{1683}{9999} = \frac{\mathbf{17}}{\mathbf{101}}.$$

The fact that the final fraction reduces so much is intriguing. (But the 17 in the numerator is probably a coincidence.) Students with sufficient number theory background might consider looking for further examples along these lines. Please share any interesting discoveries with us at `info@mandelbrot.org`.

5. Several slippery computations can be avoided by adopting the strategy of adding together the volumes of two complete triangular prisms, then subtracting out the volume of the square-based pyramid in which they intersect. Recalling the 5–12–13 right triangle, we quickly deduce that the height of the structure is 12 feet. Hence the overall volume is

$$\tfrac{1}{2}(10)(12)(30) + \tfrac{1}{2}(10)(12)(30) - \tfrac{1}{3}(10)(10)(12) = 3600 - 400 = \mathbf{3200}.$$

6. We will show in general that if $a_1 = c$, then the value of the infinite sum is $\frac{1}{c-1}$. This can be shown via a telescoping sum. Observe that

$$\frac{1}{a_n^2 - a_n} - \frac{1}{(a_n^2 - a_n)(a_n^2 - a_n + 1)} = \frac{1}{a_n^2 - a_n + 1}.$$

(Granted, this is a pretty devious step.) Using $a_{n+1} = a_n^2 - a_n + 1$, we deduce that

$$\frac{1}{a_{n+1}} = \frac{1}{a_{n+1} - 1} - \frac{1}{(a_{n+1} - 1)a_{n+1}}.$$

But the last denominator can be rewritten as $a_{n+1}^2 - a_{n+1} = a_{n+2} - 1$, so

$$\frac{1}{a_{n+1}} = \frac{1}{a_{n+1} - 1} - \frac{1}{a_{n+2} - 1}.$$

Therefore the entire sum becomes

$$\left(\frac{1}{a_1 - 1} - \frac{1}{a_2 - 1}\right) + \left(\frac{1}{a_2 - 1} - \frac{1}{a_3 - 1}\right) + \left(\frac{1}{a_3 - 1} - \frac{1}{a_4 - 1}\right) + \cdots,$$

which reduces to just $\frac{1}{c-1}$, where $a_1 = c$. We need $c > 1$ so that the terms in the sum become smaller and smaller. (Technically, $c < 0$ also works.) In our case $c = \sqrt{3}$, so the series converges to $\frac{1}{\sqrt{3}-1} = \mathbf{\frac{1}{2}(\sqrt{3} + 1)}$.

7. Since ξ is a fifth root of unity, we know that $\xi^5 = 1$, $\xi^6 = \xi$, and so forth. Furthermore, the numbers 1, ξ, ξ^2, ξ^3 and ξ^4 are the five roots of the polynomial $x^5 - 1 = 0$. Since there is no x^4 term, we deduce that $1 + \xi + \xi^2 + \xi^3 + \xi^4 = 0$, which will come in handy shortly.

For the sake of symmetry, we set $e = 20 + 13$ and instead compute the sum $a^3 + b^3 + c^3 + d^3 + e^3$. Cubing $a = 20\xi^2 + 13\xi$ gives

$$a^3 = 20^3\xi^6 + 3(20^2)(13)\xi^5 + 3(20)(13^2)\xi^4 + 13^3\xi^3.$$

The expansions for b^3, c^3, d^3 and e^3 will be similar, only with different powers of ξ. Collecting the leading terms (involving 20^3), we obtain

$$20^3(\xi^6 + \xi^{12} + \xi^9 + \xi^3 + 1) = 20^3(\xi + \xi^2 + \xi^4 + \xi^3 + 1) = 0.$$

In a similar fashion the terms involving $3(20)(13^2)$ and 13^3 also cancel. However, in each case the coefficient of $3(20^2)(13)$ is 1, leaving us with

$$a^3 + b^3 + c^3 + d^3 + e^3 = 5(3)(20^2)(13).$$

Subtracting $e^3 = 33^3$ from both sides leaves us with the desired sum

$$a^3 + b^3 + c^3 + d^3 = 5(3)(20^2)(13) - 33^3 = \mathbf{42063}.$$

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