March 2013

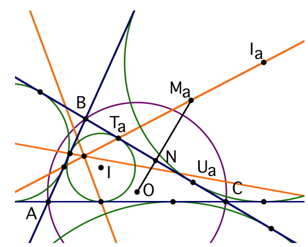
The Mandelbrot Team Play

Round Three Test

Time Limit: 60 minutes

Facts: Given three non-collinear points A, B and C in the plane, there are precisely four circles tangent to all three of the lines AB, AC and BC. The one within $\triangle ABC$ is the *incircle*, while the remaining three are called *excircles*. The centers and radii of these circles are commonly denoted I, I_a, I_b, I_c and r, r_a, r_b, r_c. There are many relationships among the lengths defined by the points of tangency, such as the fact that $BT_a = CU_a$ in the diagram.

Setup: Given a triangle ABC in the plane, let the incircle have center I and be tangent to sides $\overline{BC}$, $\overline{AC}$ and $\overline{AB}$ at T_a, T_b and T_c, respectively. Next let I_a, I_b and I_c be the centers of the three excircles. We also define M_a, M_b and M_c as the midpoints of segments I_aT_a, I_bT_b and I_cT_c. Finally, let O be the center of the circle through A, B and C. A portion of this diagram is reproduced at right. The goal of the questions below will be to establish the beautiful result that lines OI, I_aT_a, I_bT_b and I_cT_c are all concurrent.



Problems

Part i: (4 points) Let U_a be the point where side $\overline{BC}$ is tangent to the excircle around I_a, and let N be the midpoint of segment T_aU_a. Show that $\overline{ON}$ is perpendicular to $\overline{BC}$.

Part ii: (4 points) Explain why line ON intersects segment I_aT_a at its midpoint M_a.

Part iii: (5 points) Let Q be the point where $\overrightarrow{ON}$ meets the circle through A, B and C. Show that Q lies on the angle bisector of $\angle BAC$ along with I and I_a.

Part iv: (5 points) Prove that $QM_a = \frac{1}{2}r$, where r is the inradius of $\triangle ABC$. What does this fact tell us about the circle through M_a, M_b and M_c? Justify your observation.

Part v: (5 points) Prove that lines OI, I_aT_a, I_bT_b and I_cT_c are all concurrent.

Part vi: (5 points) As a bonus fact, use part iv to show that $r_a + 2R\cos A = 2R + r$, where R is the radius of the circle through A, B and C.