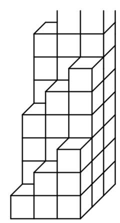
Answer Key	
1. 6	4. 24
2. 102	5. 39π
3. 53	6. 7
	7. $-1/3$

1. Suppose we use the digits 1 and 6 to obtain $1 \times 6 = 6$. Then the only pair of distinct digits left that add to 8 are $3 + 5 = 8$. Once these are used, we can write $8 - 4 = 4$ to obtain a difference of 4, but then it's impossible to get a quotient of 2. So writing $1 \times 6 = 6$ was a bad idea.

The other possibility is to write $2 \times 3 = 6$. Now we can only add a pair of distinct digits to get 8 as $1 + 7 = 8$. Next we look for a quotient of 2, which is possible with what's left by dividing $8 \div 4 = 2$. Finally, a difference of 4 is obtained via $9 - 5 = 4$, leaving only **6** left.



2. Observe that the number of unit squares facing forward is exactly $7 + 8 + 9 = 24$. (Imagine a purely front-on perspective.) Obviously the number or rear-facing squares is also 24. Similarly reasoning reveals that the number of left-facing squares is $3 + 6 + 9 = 18$, and same for right-facing squares. Finally, there are 9 squares facing up and another 9 squares facing down, for a total surface area of

$$24 + 24 + 18 + 18 + 9 + 9 = \mathbf{102}.$$

3. Since the only even prime is 2, the equation $q+r+s = 89$ implies that all three of q, r, s are odd primes. But then we deduce from $p+q+r = 72$ that p must even, hence $p = 2$. Our three equations now reduce to

$$q + r = 70, \quad r + s = 72, \quad q + r + s = 89.$$

Adding the first two and subtracting the third yields $r = 53$, from which we quickly conclude that $q = 17$ and $s = 19$. Therefore $r = \mathbf{53}$ is the largest of the four primes.

4. It will be convenient to let w represent the quantity $\log_x y$. According to Laws of Logarithms we then have $\log_y x = \frac{1}{w}$, meaning that $\log_y(x^2) = 2\log_y x = \frac{2}{w}$. Therefore the first equation gives

$$w - \frac{2}{w} = 1 \quad \Longrightarrow \quad w^2 - w - 2 = (w+1)(w-2) = 0,$$

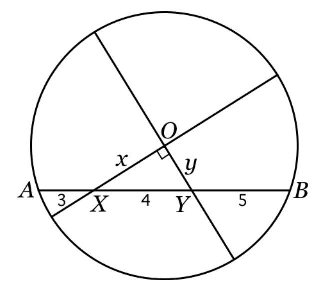
hence $w = -1$ or 2. The first case leads to $\log_x y = -1$, or $y = x^{-1} = \frac{1}{x}$. Hence the second equation gives $x + \frac{1}{x} = 20$, or $x^2 - 20x + 1 = 0$. This quadratic has two positive roots whose sum is 20, by Viete. The other possibility is that $w = 2$, which corresponds to $\log_x y = 2$, or $y = x^2$. Substituting into the second equation yields $x^2 + x = 20$, which has the roots $x = 4$ and $x = -5$. Only the first of these is positive, however, so the desired sum of all possible positive values of x is $20 + 4 = \mathbf{24}$.

5. Label the distances from X and Y to the center O as x and y, and let r be the radius of the circle. Then by power of a point at X we have

$$(r - x)(r + x) = (3)(4 + 5) \quad \Longrightarrow \quad r^2 - x^2 = 27.$$

Doing the same thing at Y gives

$$r^2 - y^2 = (5)(3 + 4) = 35.$$



Adding these equations, and using the fact that $x^2 + y^2 = 16$ by Pythagorean, we find that $2r^2 - 16 = 62$, so $r^2 = \frac{1}{2}(78) = 39$. Thus the area of the circle is $\pi r^2 = \mathbf{39}\pi$.

6. Pruning once gives 3, 5, 7, 9, 11, 13, ..., and pruning a second time yields 7, 13, 19, ... It is straightforward to prove (using induction) that every resulting sequence will be an arithmetic sequence with some first term a and common difference $a - 1$, because pruning the sequence

$$a,\ 2a - 1,\ 3a - 2,\ 4a - 3,\ \ldots$$

gives a sequence whose first term is $a(a) - (a - 1) = a^2 - a + 1$, with common difference $a(a - 1) = a^2 - a$. In fact, if we let a_n be the first term of the nth sequence, we have just shown that $a_{n+1} = a_n^2 - a_n + 1$. Since $a_0 = 2$, we now examine these first terms mod 41:

$$a_0 = 2, \quad a_1 = 3, \quad a_2 = 7, \quad a_3 = 7^2 - 7 + 1 \equiv 2 \bmod 41, \quad \ldots$$

But now the values will cycle, with $a_4 \equiv 2^2 - 2 + 1 \equiv 3 \bmod 41$, and so forth. Pruning twenty times will put us at the end of the three-term cycle, hence $a_{20} \equiv \mathbf{7} \bmod 41$.

7. Since we are interested in finding $a+b+c$, let us call this quantity X. Thus $(a+b)$ can be written as $(X-c)$, and so forth. Therefore the main equation may be rewritten as

$$(X-c)(X-b)(X-a) = X^2\left(3 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

By the givens, we also have $X = abc$, so the righthand side expands to $3X^2 + X(ab+ac+bc)$. Expanding the lefthand side as well yields

$$X^3 - X^2(a+b+c) + X(ab+ac+bc) - abc = 3X^2 + X(ab+ac+bc).$$

Since $X = a+b+c$ the first two terms on the left cancel, so this boils down to $-X = 3X^2$. But $X \neq 0$, so dividing brings us to $X = -\mathbf{1/3}$. In fact there are infinitely many triples (a,b,c) satisfying these equations, such as $a = \frac{1}{3}(-2+\sqrt{7})$, $b = \frac{1}{3}(-2-\sqrt{7})$, $c = 1$.

Problem Credits

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