Answer Key	
1. 9	4. 81
2. 3.5	5. -246
3. 27	6. $\sqrt{7}-1$
	7. $5/8$

1. We could follow Claire's lead and add up all the two-digit numbers ending in 3, 1, 4, 5 or 9, but there is a better way. Consider only such numbers from 60 to 69, for instance. Their sum is

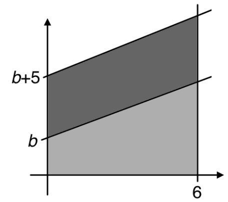
$$63+61+64+65+69=300+(3+1+4+5+9)=300+22.$$

The remaining numbers from 60 to 69 have sum

$$60+62+66+67+68=300+(0+2+6+7+8)=300+23,$$

so the second sum is 1 greater. Clearly the same will be true for each of the nine possible tens digits, so in the end the second sum will **9** greater.

2. The regions bordered by the lines is shown at right. The lower region is a trapezoid with base of length 6. To find the heights, we plug in $x=0$ and $x=6$ to the equation of the line to obtain b and $b+3$. Hence its area is $\frac{1}{2}(6)(b+b+3)=6b+9$. The upper region is a parallelogram, also with base 6 but with height 5, hence its area is $(6)(5)=30$. So we want $6b+9=30$, which gives $b=21/6=\mathbf{3.5}$.



3. It will help to figure out that

$$\text{LCM}(1,2,3,\ldots,50)=2^5\cdot 3^3\cdot 5^2\cdot 7^2\cdot 11\cdot 13\cdots 47,$$

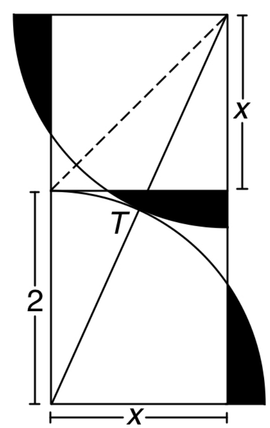
where all the primes from 11 to 47 appear to the first power. We next notice that $n=30$ is too large, since $\text{LCM}(30,31,32,\ldots,50)$ does not have a factor of 29. But $\text{LCM}(29,30,31,\ldots,50)$ will at least include every prime. (Note that 23 appears due to the 46.) However, we have to worry about prime powers. And while $n=29$ does involve a 2^5 in the LCM, it only includes a 3^2. We must back up to $n=27$ to obtain all the primes to the correct powers, so **27** is the largest value.

4. To minimize the amount of casework necessary, we first consider the placement of the digit 5. Clearly it cannot appear as the final digit of any number, otherwise either another 5 or a 0 would be required. It also can't be the tens digit of the second factor in the bottom line. (See why?) So it must be the tens digit of one of the products, leading us to try out $6\times 9=54$ or $7\times 8=56$ for the top line, or $4\times 13=52$, $3\times 18=54$ or $3\times 19=57$ in the bottom line. The first of these options leads to the unique solution, $6\times 9=54$ and $3\times 27=\mathbf{81}$.

5. We know $p(0)$ is equal to the constant term of $p(x)$, which is the negative product of its roots; namely, $-(\alpha+\beta)(\alpha+\gamma)(\beta+\gamma)$. This expression is symmetric, so we must be able to write it in terms of elementary symmetric functions. A bit of experimentation leads us to

$$-(\alpha+\beta+\gamma)(\alpha\beta+\alpha\gamma+\beta\gamma)+\alpha\beta\gamma=-(-20)(-13)+14=\mathbf{-246}.$$

6. Here is an elegant approach that avoids coordinates, trigonometry, and messy algebra. Extend the circular arc in the upper right corner of the rectangle to obtain another quarter circle above the original one, as shown below. We now make two crucial observations. First, the diagram is now symmetric, meaning that a half-turn about some point carries each quarter circle onto the other. Since the point of tangency T is the only point common to both circles, it must be the center of symmetry. This implies, for example, that the main diagonal passes through T and has length 4.



Secondly, the upper quarter circle is also symmetric over the dotted diagonal, which implies that the portion of the rectangle above the horizontal line must be a square. In particular, its height is x, as labeled. Applying the Pythagorean Theorem to the right triangle formed by the main diagonal reveals that

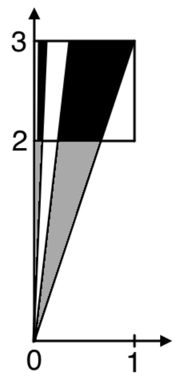
$$x^2 + (2+x)^2 = 4^2 \qquad \Longrightarrow \qquad x = \sqrt{\mathbf{7}} - \mathbf{1},$$

via a straightforward application of the quadratic formula.

7. We are looking for pairs (x, y) with $0 < x < 1$ and $2 < y < 3$ for which $\lfloor \log_3(\frac{y}{x}) \rfloor$ is odd. Since $\frac{y}{x} > 2$ for any such pair, this means that x and y must satisfy one of

$$1 \le \log_3\left(\frac{y}{x}\right) < 2, \quad 3 \le \log_3\left(\frac{y}{x}\right) < 4, \quad 5 \le \log_3\left(\frac{y}{x}\right) < 6, \quad \cdots.$$

Hence the slope $m = \frac{y}{x}$ of the line joining $(0,0)$ to (x, y) must lie in one of the ranges $3 \le m < 9$ or $27 \le m < 81$, and so forth. This leads to the diagram shown at right, where the shaded regions correspond to pairs (x, y) for which $\lfloor \log_3(\frac{y}{x}) \rfloor$ is odd. Since the area of the square $0 < x < 1$ and $2 < y < 3$ is equal to 1, the probability is just the area of the dark shaded region. Rather than deal with trapezoids, we find the area of all the triangles, including the light shaded regions also, and then scale by 5/9 at the end. The triangles have total area



$$\frac{1}{2}(3)\left(\frac{2}{3} + \frac{2}{27} + \frac{2}{243} + \cdots\right) = \frac{3}{2} \cdot \frac{\frac{2}{3}}{1 - \frac{1}{9}} = \frac{3}{2} \cdot \frac{3}{4} = \frac{9}{8}.$$

Hence the area of the dark shaded regions is $(\frac{5}{9})(\frac{9}{8}) = \mathbf{\frac{5}{8}}$, which is the desired probability.

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⋆ National Level ⋆

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PO Box 760
Potsdam, NY 13676

The Mandelbrot Competition
www.mandelbrot.org
info@mandelbrot.org