March 2012

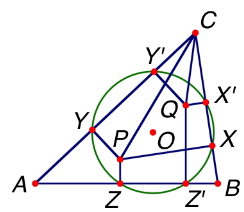
The Mandelbrot Team Play

Round Three Test

Time Limit: 60 minutes

Facts: The *Theorem on Cyclic Quadrilaterals* states that if points C and D are situated on the same side of line AB then $\angle ACB \cong \angle ADB$ exactly when the four points lie on a single circle. Similarly, if C and D are on opposite sides of line AB, then angles $\angle ACB$ and $\angle ADB$ are supplementary if and only if the points lie on a circle. The *Power of a Point Theorem* asserts that if two lines through a given point P intersect a circle at pairs of points A, B and C, D then we have $(PA)(PB) = (PC)(PD)$.

Setup: Let P be any point within triangle ABC. Drop perpendiculars from P meeting lines BC, AC and AB at points X, Y and Z, respectively. Next draw the circle through X, Y and Z. Label its center as O, and suppose that it intersects lines BC, AC and AB a second time at points X', Y' and Z', as shown. We will demonstrate the remarkable fact that the perpendiculars through X', Y' and Z' all meet at a single point Q, which turns out to be the *isogonal conjugate* of P, meaning that $\angle ACP \cong \angle BCQ$ and similarly.



Problems

Part i: (4 points) Explain why the perpendicular from O to $\overline{AB}$ bisects $\overline{ZZ'}$.

Part ii: (5 points) Now let Q be the point such that O is the midpoint of $\overline{PQ}$. Prove that the line through Z' perpendicular to $\overline{AB}$ passes through Q. (You may assume without proof that by the same reasoning the perpendiculars through X' and Y' also pass through Q.)

Part iii: (4 points) Next show that $\angle ACP \cong \angle YXP$ and that $\angle BCQ \cong \angle X'Y'Q$. (In this part and the next you may assume that Q lies inside triangle ABC.)

Part iv: (5 points) Prove that $\angle ACP \cong \angle BCQ$ by establishing that $\angle YXP \cong \angle X'Y'Q$.

Part v: (5 points) Suppose that instead of dropping perpendiculars, we let X, Y and Z be the points where lines AP, BP and CP intersect the opposite sides of $\triangle ABC$. Then define X', Y' and Z' in the same manner as above. Prove that AX', BY' and CZ' are concurrent.

Part vi: (5 points) Demonstrate that any time points X, X', Y, Y', Z and Z' on the sides of $\triangle ABC$ lie on a single circle, the quantities $(AY + AY')^2 + (BZ + BZ')^2 + (CX + CX')^2$ and $(AZ + AZ')^2 + (BX + BX')^2 + (CY + CY')^2$ are equal.