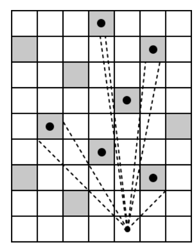
Answer Key	
1. 24	4. 2/3
2. 7	5. 42
3. 16	6. 1513
	7. 105

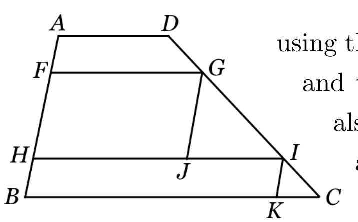


1. One convenient way to organize our work is to consider all shaded squares above the central dotted square, between lines with slope ± 1, as shown in the diagram. By carefully sweeping out this region from one side to the other, we see that exactly six shaded squares are visible from the central dot, as marked. By symmetry, the same thing occurs to the right, left, and below the central dot, for a total of **24** visible shaded squares.

2. Clearly it requires two turns to reach 4. From here it is tempting to multiply our way up to 64, then subtract 2 twice to obtain 60, in eight turns. But it's possible to improve on this strategy by multiplying up to 32 in five turns, then subtracting 2, and finally multiplying by 2 to get to 60 in only **7** turns.

3. We present a purely geometric approach. First draw parallels to $\overline{AB}$ through F and H. Because of the parallelograms formed, we have $FG = HJ$ and $HI = BK$, for instance. Now observe that the difference between the perimeters of $ABCD$ and $AHID$ is

$$(AB + BC + CD + DA) - (AH + HI + ID + DA) = HB + KC + IC,$$



using the parallelograms. According to the givens and the similar triangles $\triangle IKC \sim \triangle GJI$, we also see that $HB = \frac{1}{2}(HF)$, $KC = \frac{1}{2}(JI)$, and $IC = \frac{1}{2}(GI)$. Hence the difference in perimeters is precisely $\frac{1}{2}(HF + JI + GI)$, which matches half the difference between the perimeters of $AFGD$ and $AHID$. In other words, perimeter$(ABCD) = 14 + \frac{1}{2}(14 - 10) = \mathbf{16}$.

4. Finding the probability that *none* of the numbers $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\ldots$ lie between r and s turns out to be more accessible, so we tackle this question instead. For this to occur we must have both r and s between $\frac{1}{2}$ and 1, or both between $\frac{1}{4}$ and $\frac{1}{2}$, or both between $\frac{1}{8}$ and $\frac{1}{4}$, and so on. This total probability is given by

$$(\tfrac{1}{2})(\tfrac{1}{2}) + (\tfrac{1}{4})(\tfrac{1}{4}) + (\tfrac{1}{8})(\tfrac{1}{8}) + \cdots = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{1}{3},$$

since this a geometric series with first term $\frac{1}{4}$ and common ratio $\frac{1}{4}$ as well. Hence our desired probability is $1 - \frac{1}{3} = \mathbf{\frac{2}{3}}$.

5. To begin, let's determine how many complex numbers are solutions to both $z^{60} = 1$ and $z^{72} = 1$. The roots of these equations will be 60 (or 72) points evenly spaced around the unit circle, starting with 1. The first "interesting" point in common is $\frac{5}{60} = \frac{6}{72} = \frac{1}{12}$ of the way around, so there will be 12 points in common all together. In other words, the points in common will be precisely the $\mathrm{GCD}(60, 72) = 12$ equally spaced points around the circle.

Extending this reasoning to the other pairs of equations, we have $\mathrm{GCD}(60, 90) = 30$ common solutions for the first and third equations, and $\mathrm{GCD}(72, 90) = 18$ common solutions to the second and third. Furthermore, the GCD of all three numbers is 6, which means we must subtract this from each pair's total to avoid counting solutions to all three equations. Our final answer is

$$(12 - 6) + (30 - 6) + (18 - 6) = \mathbf{42}.$$

6. We first rewrite each factor as

$$\left(\sqrt{k} + \frac{1}{\sqrt{k}}\right) = \sqrt{k}\left(1 + \frac{1}{k}\right) = \sqrt{k}\left(\frac{k+1}{k}\right).$$

It now becomes clear that the entire product telescopes to

$$P = \sqrt{2 \cdot 3 \cdots 2015} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdots \frac{2016}{2015} = \sqrt{2015!} \cdot 1008.$$

But 2017 is prime, so by Wilson's Theorem we have $2015! \equiv 1 \bmod 2017$. This means $P^2 \equiv 1008^2 \bmod 2017$, or

$$4P^2 \equiv 2016^2 \equiv (-1)^2 \equiv 1 \bmod 2017.$$

Note $2016 = 4(504)$, so multiplying by 504 gives $-P^2 \equiv 504 \bmod 2017$, or $P^2 \equiv \mathbf{1513} \bmod 2017$.

7. With a bit of arithmetic insight, one realizes that it will be advantageous to arrange all the numbers in an 11×11 square grid, with the numbers from 1-11 along the top row, 13-23 along the second row, down to 121-131 along the bottom row. The desired sets of three numbers, forming arithmetic sequences with difference 1 or 12, are neatly situated as horizontal or vertical triplets. In fact, it now becomes clear that finding such a partition is equivalent to tiling this 11×11 grid with forty 3×1 rectangles, and asking which numbered square remains uncovered.

As we shall see, the remaining square can only be in row 3, 6 or 9 and same for the column. The highest numbered such square, in row 9 and column 9, is given by $12(8) + 9 = \mathbf{105}$, the desired number. Here is a brief explanation for why those nine squares are the only possibilities. On the 11×11 grid, color the diagonals with slope 1 alternately red, blue, green, red, blue, green, and so forth. One can verify that this results in 40 red squares, 41 blue squares, and 40 green squares. Any tiling by 3×1 rectangles will cover exactly forty of each color, leaving a blue square left over. The same is true for diagonals with slope -1; the only squares common to both sets of blue diagonals are the ones described above, and you're done.

Problem Credits

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⋆ National Level ⋆

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