Team Play Solutions

Part i: Iterating the process beginning with $a_1 = 3$, $b_1 = 7$ yields

$$(3,7) \;\to\; (3,4) \;\to\; (3,1) \;\to\; (1,2) \;\to\; (1,1) \;\to\; (1,0) \;\to\; (0,1) \;\to\; \cdots,$$

at which point the sequence of pairs stabilizes. Adding together pairs of products gives $21 + 12 + 3 + 2 + 1 + 0 + \cdots = 39$. When we begin with $(10, 17)$ instead the next couple of pairs are $(10, 7)$ and $(7, 3)$, at which point we get the same thing as above. So to compute $g(10, 17)$ we need only add on the products $10 \cdot 17$ and $10 \cdot 7$ to our previous total to obtain $170 + 70 + 39 = 279$. Finally, the value of $g(5, 100)$ is

$$5 \cdot 100 + 5 \cdot 95 + 5 \cdot 90 + \cdots + 5 \cdot 5 = 5(20)(52.5) = 5250,$$

using the fact that $100 + 95 + \cdots + 10 + 5$ has 20 terms, each of which is $\frac{1}{2}(100 + 5)$ on average.

Part ii: Beginning with $a_1 = 2$, $b_1 = 2k$ leads to

$$(2, 2k) \;\to\; (2, 2k-2) \;\to\; \cdots \;\to\; (2,2) \;\to\; (2,0) \;\to\; (0,2) \;\to\; \cdots$$

Therefore the value of $g(2, 2k)$ is given by

$$2(2k) + 2(2k-2) + \cdots + 2(2) = 2(k)(k+1) = 2k^2 + 2k,$$

as claimed. In the same fashion, starting with $a_1 = 2$, $b_1 = 2k + 1$ gives

$$(2, 2k+1) \;\to\; (2, 2k-1) \;\to\; \cdots \;\to\; (2,1) \;\to\; (1,1) \;\to\; (1,0) \;\to\; \cdots$$

Using our technique for summing terms of an arithmetic sequence,

$$(2k+1) + (2k-1) + \cdots + 1 = (k+1)(k+1) = k^2 + 2k + 1.$$

Hence $g(2, 2k+1) = 2(k^2 + 2k + 1) + 1 = 2k^2 + 4k + 3$.

Part iii: Clearly if we add the smaller of two positive real numbers to their positive difference, then we obtain the larger one. (Take a moment to convince yourself of this fact.) In our situation, this means that $(a_2 + b_2)$ is equal to the larger of a_1 and b_1. Since a_2 is defined as the smaller of the two, we deduce that $a_2 + (a_2 + b_2) = a_1 + b_1$. The same is true for each subsequent pair of numbers, so we know that

$$\begin{aligned}
a_2 + (a_2 + b_2) &= a_1 + b_1, \\
a_3 + (a_3 + b_3) &= a_2 + b_2, \\
&\vdots \\
a_n + (a_n + b_n) &= a_{n-1} + b_{n-1}.
\end{aligned}$$

Adding these equalities, cancelling terms, and using the fact that $a_1 = x$, $b_1 = y$ gives $(a_2 + a_3 + \cdots + a_n) + (a_n + b_n) = x + y$, as desired. Finally, since $(a_n + b_n)$ is always positive, we know that $a_2 + a_3 + \cdots + a_n < x + y$ for any n. Adding together infinitely many terms could make the sum arbitrarily close to $x + y$, though, so in the limit at best we can conclude that $a_2 + a_3 + a_4 + \cdots \le x + y$.

Part iv: Suppose that $a_1, b_1 < \epsilon$. Clearly both the smaller of a_1, b_1 as well as the positive difference between a_1, b_1 will also be less than ϵ. In other words, $a_2, b_2 < \epsilon$. Repeating this argument shows that $a_3, b_3 < \epsilon$, and so forth. Now use the definition of $g(x, y)$, the fact that $\epsilon < 1$, and the result of the previous part to deduce that

$$\begin{aligned}
g(x,y) &= a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4 + \cdots \\
&< \epsilon(a_1 + a_2 + a_3 + a_4 + \cdots) \\
&< a_1 + a_2 + a_3 + a_4 + \cdots \\
&< \epsilon + x + y \\
&< 3\epsilon.
\end{aligned}$$

Part v: We will show that for any $n \ge 1$ we have

$$g(a_n, b_n) - \tfrac{1}{2}(a_n^2 + a_n b_n + b_n^2) = g(a_{n+1}, b_{n+1}) - \tfrac{1}{2}(a_{n+1}^2 + a_{n+1} b_{n+1} + b_{n+1}^2),$$

which will prove the claim. Rearranging, this is equivalent to

$$g(a_n, b_n) - g(a_{n+1}, b_{n+1}) = \tfrac{1}{2}(a_n^2 + a_n b_n + b_n^2) - \tfrac{1}{2}(a_{n+1}^2 + a_{n+1} b_{n+1} + b_{n+1}^2).$$

We first observe that the left-hand side is equal to $a_n b_n$, by the definition of $g(x, y)$. (The sum for computing $g(a_n, b_n)$ has one more term than that for $g(a_{n+1}, b_{n+1})$, and all but that term cancel when subtracting.)

Now consider the right-hand side. Since it is symmetric in a_n, b_n we can assume without loss of generality that $a_n \leq b_n$. It follows that $a_{n+1} = \min(a_n, b_n) = a_n$ and $b_{n+1} = |a_n - b_n| = b_n - a_n$. Therefore

$$\begin{aligned} a_{n+1}^2 + a_{n+1}b_{n+1} + b_{n+1}^2 &= a_n^2 + a_n(b_n - a_n) + (b_n - a_n)^2 \\ &= a_n^2 + a_n b_n - a_n^2 + b_n^2 - 2a_n b_n + a_n^2 \\ &= a_n^2 - a_n b_n + b_n^2. \end{aligned}$$

Hence the entire right-hand side above reduces to just $a_n b_n$ also.

Part vi: To begin, notice that by the nature of how each pair of numbers is formed from the previous, if we begin with $a_1 = \pi$ and $b_1 = 1$, then every subsequent value of a_n and b_n will be of the form $c + d\pi$ for integers c and d. For instance, the next several pairs after $(\pi, 1)$ are

$$(1, \pi - 1) \;\rightarrow\; (1, \pi - 2) \;\rightarrow\; (1, \pi - 3) \;\rightarrow\; (\pi - 3, 4 - \pi) \;\rightarrow\; (\pi - 3, 7 - 2\pi).$$

Since π is irrational, no value of a_n or b_n will ever equal 0.

But these values do become arbitrarily small. In fact, we claim that after a finite number of steps the larger value will become only half as large as before. To see why, note that if $a_n < b_n$ then we will repeatedly subtract a_n from b_n in the next couple of iterations, until the result is less than a_n. If we happened to have $a_n < \frac{1}{2}b_n$, then we are now done. Otherwise we had $\frac{1}{2}b_n < a_n < b_n$, so the first positive difference was less than $\frac{1}{2}b_n$, and this will eventually become the larger value.

The upshot is that when $a_1 = \pi$ and $b_1 = 1$ the values of a_n and b_n are always positive, but become arbitrarily small. According to part iv, this means that $g(a_n, b_n)$ also becomes arbitrarily small. And clearly so will the value of $\frac{1}{2}(a_n^2 + a_n b_n + b_n^2)$. Therefore $g(a_n, b_n) - \frac{1}{2}(a_n^2 + a_n b_n + b_n^2)$ gets closer and closer to 0 as we iterate this process. But this value is also constant, which forces it to be equal to 0. Therefore it was 0 at the very first step, so $g(\pi, 1) = \frac{1}{2}(\pi^2 + \pi + 1)$.

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Round Two Solutions

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