January 2014

The Mandelbrot Team Play

Round One Test

Time Limit: 60 minutes

Facts: The *minimum function* min returns the smallest of two or more real numbers, giving the common least value in case there is a tie. Thus $\min(\pi, \sqrt{10}) = \pi$, $\min(-7, -8, -9) = -9$, and $\min(14, 20, 14) = 14$. The graph of $\min(2x, 6)$ is a *continuous, piecewise linear function*, meaning that in the vicinity of each x-value the graph is a line, except at *nonlinear points*, where two lines with different slopes meet. Thus $\min(2x, 6)$ has a nonlinear point at $x = 3$.

Setup: In *tropical math* there are two operations $\oplus$ and $\odot$, defined as $x \oplus y = \min(x, y)$ and $x \odot y = x + y$. We use the symbols $\oplus$ and $\odot$ since this pair of operations bears many similarities to ordinary addition and multiplication. As with standard order of operations, $\odot$ is evaluated first, followed by $\oplus$. For example, $5 \oplus 4 \odot 3 = 5 \oplus 7 = 5$. To graph a function such as $x \odot x \oplus 3 \odot x \oplus 5$, sketch the graphs of $y = x + x$, $y = x + 3$ and $y = 5$, then draw $\min(2x, x + 3, 5)$ by taking the lower boundary of these three graphs. Finally, the *roots* of a tropical polynomial are defined as the x-coordinates of the nonlinear points on its graph.

Problems

Part i: (4 points) Draw a detailed graph of $y = 3 \odot x \odot x \oplus 5 \odot x \oplus 12$, including coordinates of the nonlinear points. For what value of x does $3 \odot x \odot x \oplus 5 \odot x \oplus 12$ equal 6?

Part ii: (4 points) Why does it make sense algebraically that expanding $(x \oplus 4) \odot (x \oplus 7)$ would give $x \odot x \oplus 4 \odot x \oplus 11$? In a similar manner, expand $(x \oplus 20) \odot (x \oplus 14)$.

Part iii: (5 points) Consider the tropical polynomial $x \odot x \odot x \oplus 3 \odot x \odot x \oplus c \odot x \oplus 27$. Find the three distinct roots of this polynomial when $c = 8$. For what values of c does the resulting polynomial have only two distinct roots?

Part iv: (5 points) Demonstrate that the definition of roots given above is reasonable by proving that for any numbers a and b, the roots of $(x \oplus a) \odot (x \oplus b)$ are $x = a$ and $x = b$.

Part v: (5 points) The three-dimensional graph of $5 \odot x \oplus 7 \odot y \oplus 10$ has nonlinear points, where different planes meet. Its *zero set* consists of all pairs (x, y) of x and y-coordinates of nonlinear points. Draw the graph of this zero set (in the xy-plane), justifying your answer.

Part vi: (5 points) For real numbers a and b the zero set of $a \odot x \oplus b \odot y \oplus 0$ is called a tropical line. Given two points in the plane, prove that in general there is a unique tropical line passing through both points, and describe the exceptions.