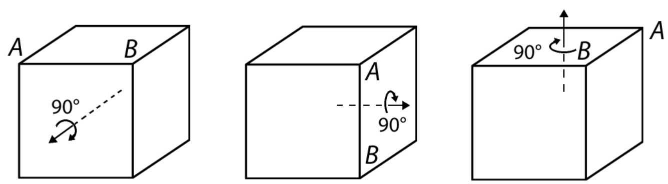
Answer Key

1. 17	4. \$31
2. 3	5. $2\sqrt{13}$
3. 13	6. $34\sqrt{17}$
	7. 268

1. This linear equation is straight-forward to solve in the usual manner. However, it is also possible to solve the equation by inspection. Observe that because of the symmetry of the two sides of the equation, plugging in $x = 17$ gives $7 \cdot 10 = 70$ on both sides. Therefore $x = \mathbf{17}$ is the answer.

2. A sequence of three quarter-turns is the least number possible. There are several ways to accomplish this task; one way is illustrated below. After the third turn vertices A and B will have reversed positions.



3. We quickly discover that 14 is nimble, since $14 = 27 - 13$, and $27 = 3^3$ while 13 is prime. At first it would seem that 13 cannot be expressed as a cube minus a prime, but the persistent solver finally discovers that $13 = 512 - 499$, and $512 = 8^3$ while 499 is prime. Hence the next nimble number is in fact **13**. It seems likely that every positive integer is nimble, although a proof of this fact is probably not easy!

4. Let x represent the price Abby will charge for the new cheaper model, so that the more expensive type will cost $x + 8$ dollars. Also let w stand for the number of widgets sold, which does not change after the new pricing structure is put in place. Abby's revenue prior to the change is \$$30w$, while it becomes

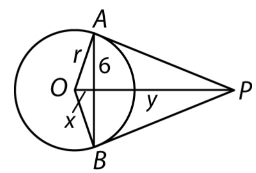
$$\left(\frac{3}{4}w\right)x + \left(\frac{1}{4}w\right)(x+8) = w(x+2)$$

afterwards, since three-quarters of the sales are for \$$x$ each, while a quarter of them sell for \$$(x + 8)$. Since a 10% increase means that the new revenue is $\frac{110}{100}$ times the old revenue, we have

$$\frac{110}{100}(30w) = w(x+2) \qquad \Longrightarrow \qquad 33 = x + 2,$$

thus the less expensive widget should cost **\$31**.

5. There are many routes to the answer. One method is note that $\overline{AB}$ is bisected by $\overline{OP}$ into two halves of length 6. On the other hand, $\overline{AB}$ cuts $\overline{OP}$ into two segments whose lengths we label as x and y. Next observe that $\triangle OAP$ is a right triangle since the line is tangent to the circle. Standard properties of altitudes in right triangles now imply that $xy = 6^2$. (Because the two smaller right triangles within $\triangle OAP$ are similar.) But $x + y = 13$, and solving these equations yields $x = 4$ or 9. Hence $r = 2\sqrt{13}$ or $r = 3\sqrt{13}$. We discard the latter solution since $r < 10$, leaving $r = \mathbf{2\sqrt{13}}$.



6. A clever application of the AM-GM (Arithmetic Mean-Geometric Mean) inequality neatly reveals the maximal value. Recall that AM-GM for three positive real numbers states that $\frac{1}{3}(x + y + z) \geq \sqrt[3]{xyz}$, with equality if and only if $x = y = z$. Taking $x = \frac{1}{2}a$, $y = \frac{1}{2}a$ and $z = 51 - a$ we discover that

$$\frac{\frac{1}{2}a + \frac{1}{2}a + (51-a)}{3} \geq \sqrt[3]{(\tfrac{1}{2}a)(\tfrac{1}{2}a)(51-a)},$$

which may be rewritten as

$$4 \cdot 17^3 \geq a^2(51-a)$$

after simplify the left side, cubing both sides, then multiplying through by 4. Taking the square root of both sides gives $a\sqrt{51-a} \leq 34\sqrt{17}$. Hence the largest value in the list occurs when $a = 34$, and the actual maximum value is $\mathbf{34\sqrt{17}}$. (Note that it is also possible to find the answer without too much difficulty using calculus.)

7. Label lengths PA, PB and PC as a, b and c, respectively. The Law of Cosines applied to $\triangle PAB$ gives $(AB)^2 = a^2 + b^2 - ab$, since $\cos 60^\circ = \frac{1}{2}$. Similarly $(AC)^2 = a^2 + c^2 - ac$ and $(BC)^2 = b^2 + c^2 - bc$. Now the Pythagorean Theorem in $\triangle ABC$ tells us that

$$(a^2 + b^2 - ab) = (a^2 + c^2 - ac) + (b^2 + c^2 - bc),$$

which may be rearranged to give

$$ac + bc - ab - c^2 = c^2.$$

Factoring, we arrive at $(b - c)(c - a) = c^2$.

We are given that $b = 2010$ and that c is odd, and wish to find a. Let d be the greatest common divisor of a, b and c, so that we can write $a = da'$, $b = db'$ and $c = dc'$. Substituting in and dividing through by d^2 gives $(b' - c')(c' - a') = (c')^2$. But now $(b' - c')$ and $(c' - a')$ are relatively prime and their product is a square, so we must have $b' - c' = m^2$, $c' - a' = n^2$ and $c = mn$. Solving for a' and b' brings us to

$$a' = n(m - n), \quad b' = m(m + n), \quad c' = mn,$$

with $m > n$. Recall that $b' \mid 2010$ and c' is odd, meaning that both m and n are odd. So b' is a factor of 2010 which can be written as the product of an odd number m and a larger even number $m + n$ which is less than twice as large as m. Since $2010 = (2)(3)(5)(67)$ there are only a couple options to consider, and only one that works, namely $m = 5$, $n = 1$ which yields $a' = 4$, $b' = 30$, $c' = 5$, and $d = 67$. Finally, we deduce that $a = 4 \cdot 67 = \mathbf{268}$.

December

⋆ NATIONAL LEVEL ⋆

The Mandelbrot Competition

Round Two Solutions

Greater Testing Concepts
PO Box 760
Potsdam, NY 13676

The Mandelbrot Competition
www.mandelbrot.org
info@mandelbrot.org