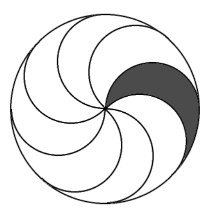
Answer Key		4.	20π
1.	PEA	5.	$\log_2 21$
2.	$8/15$	6.	$\sqrt{34}$
3.	$2/5$	7.	819

1. Of the four letters mentioned—A, E, P, R—the letter that is *not* in Brenna's word must be part of each guess, since exactly one letter in each guess is wrong. Hence this letter is R, so Brenna's word contains the letters A, E, P. Only one ordering of the letters A, E, P results in at least one letter in the right position each time; namely, **PEA**.

2. Let Indari's glass have volume V cm^3, so that she has $\frac{1}{3}V$ cm^3 of water in her glass. Since Niraek has $\frac{4}{5}$ this amount of water, he must have $\frac{4}{5} \cdot \frac{1}{3}V = \frac{4}{15}V$ cm^3 of water. Finally, his glass has twice this volume, so his glass must hold $\frac{8}{15}V$ cm^3. Therefore the desired ratio is $\mathbf{8/15}$.

3. This question illustrates the concept of *conditional probability.* There is a formula to compute such a probability that the reader is invited to look up, if desired. We opt for an explanation from first principles. There are eight card faces that Nick could have seen (two per card), three of which are black and five of which are red. We are only interested in the red faces. Of these five, two belong to the card that is red on both sides. Therefore the probability that Nick has chosen this card is $\mathbf{2/5}$.



4. The shaded region is bounded by three circular arcs. The two arcs within the large circle are semicircles with diameter 16mm, so each of these has length $\frac{1}{2}(16\pi)$ mm, which combine for 16π mm. The remaining portion of the boundary consists of an eighth of the circumference of the large circle, so it has length $\frac{1}{8}(32\pi) = 4\pi$ mm. Hence the total perimeter is $16\pi + 4\pi = \mathbf{20\pi}$ mm.

5. The given expression has the form $AB + A + B$, which reminds us of the product $(A+1)(B+1) = AB + A + B + 1$. Therefore we write

$$\begin{aligned}(\log_2 3)(\log_6 7) + (\log_2 3) + (\log_6 7) \\ &= (\log_2 3 + 1)(\log_6 7 + 1) - 1 \\ &= (\log_2 3 + \log_2 2)(\log_6 7 + \log_6 6) - 1 \\ &= (\log_2 6)(\log_6 42) - 1 \\ &= (\log_2 42) - \log_2 2 \\ &= \mathbf{\log_2 21}.\end{aligned}$$

6. The underlying principle behind this problem states that given several similar shapes, the area of each shape is proportional to the square of a particular length within that figure. For instance, in the given diagram, the semiperimeter s of $\triangle ADC$ is some multiple λ of its inradius 5. (The exact value of λ is not relevant.) Therefore

$$\mathit{area}(ADC) = 5s = 5(5\lambda) = 25\lambda.$$

But triangles $\triangle ADC$ and $\triangle CDB$ are similar with constant of proportionality $\frac{3}{5}$. Hence the semiperimeter of $\triangle CDB$ is $\frac{3}{5}s = 3\lambda$, so its area is 9λ. In the same manner, since $\triangle ABC$ is similar to both of the other triangles, its area will be λr^2, where r is the inradius. But the sum of the smaller areas equals the larger area, giving

$$25\lambda + 9\lambda = \lambda r^2 \qquad \Longrightarrow \qquad r = \mathbf{\sqrt{34}}.$$

7. We begin by attempting to include a power of 3 in the factorization of n. It doesn't make sense to use 3^1, because then $\theta(n)$ has a factor of $3^2 - 1 = 8$, which has no odd factors. (So it doesn't help make $\theta(n)$ divisible by other odd primes.) Therefore we take 3^2, which gives a factor of $3^3 - 1 = 26$. This permits us to include a factor of 13 with n, making $\theta(n)$ divisible by $13^2 - 1 = 168 = 3 \cdot 7 \cdot 8$. Finally, if we include a factor of 7 in n then $7^2 - 1 = 3 \cdot 16$ divides $\theta(n)$, providing the final necessary factor of 3. In summary, taking $n = 3^2 \cdot 7 \cdot 13 = \mathbf{819}$ gives $\theta(n) = 26 \cdot 48 \cdot 168$, which is divisible by n.

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⋆ Regional Level ⋆

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