Team Play Topics

ROUND ONE

The first section of the Round One Mandelbrot Team Play is reproduced below. A list of topics and practice problems are also provided to aid in preparation. Note that these problems are not meant to serve as a precise indicator of the problems that will appear on the contest. However, students who understand how to solve them should be able to make significantly more progress than they might have otherwise. So work hard on the problems, and good luck on the Team Play!

Facts: given a sequence a_0, a_1, a_2, a_3, a_4, $\ldots$, of numbers one can use *finite differences* to find a polynomial formula for the sequence. Compute the differences $b_0 = a_1 - a_0$, $b_1 = a_2 - a_1$, $b_2 = a_3 - a_2$, and so forth, writing these values in a row beneath the first. Next compute a second row of differences $c_0 = b_1 - b_0$, $c_1 = b_2 - b_1$, $c_2 = b_3 - b_2$, $\ldots$, and continue this process until some row contains all 0's. Then the desired formula is

$$a_n = a_0 + b_0 n + c_0 \binom{n}{2} + d_0 \binom{n}{3} + \cdots = a_0 + b_0 n + c_0 \frac{n(n-1)}{2\cdot 1} + d_0 \frac{n(n-1)(n-2)}{3\cdot 2\cdot 1} + \cdots.$$

Note that if the sequence begins at a_1 or later then one should extend the terms of the rows backwards, working from bottom to top, to obtain the values of a_0, b_0, c_0, d_0, etc.

TOPICS: divisibility, polynomial long division, finite differences

Practice Problems

In this set of practice problems we will look for positive integers a and b for which $\dfrac{a^3+b^3}{ab+1}$ is also a positive integer. We will call such a pair (a, b) an *excellent* pair of numbers.

1. Show that the pair $(1, b)$ is excellent for any positive integer b.

2. Prove that the only excellent pair with $a = b$ is $(1, 1)$.

3. Find several excellent pairs of numbers with $1 < a < b$ by hand. Make and prove a conjecture based on your results.

4. Find a polynomial giving values of 1, 2, 9, 25, 53, 96 when we plug in $x = 1, 2, 3, 4, 5, 6$.

5. A computer search for excellent pairs turns up the following numbers:

$$(2,4)\ (2,10)\ (2,31)\ (3,9)\ (3,17)\ (3,30)\ (4,11)\ (4,16)\ (4,26)\ (4,29)\ (4,68)$$
$$(5,7)\ (5,11)\ (5,25)\ (5,37)\ (6,36)\ (6,50)\ (7,49)\ (7,65)\ (8,64)\ (8,82).$$

Find at least two infinite families of excellent pairs of numbers, then prove your assertions.

Hints and answers on the next page. $\Longrightarrow$

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Hints and Answers

1. When $a = 1$ the expression reduces to $\dfrac{b^3+1}{b+1} = b^2 - b + 1$, which is always a positive integer.

2. When $a = b$ the given expression may be rewritten as $\dfrac{2a^3}{a^2+1} = 2a - \dfrac{2a}{a^2+1}$. But it is easy to show that $a^2 + 1 > 2a$ whenever $a > 1$ (for instance, note that $(a-1)^2 > 0$ when $a > 1$), so the resulting expression will never be an integer.

3. A bit of trial and error reveals that the pairs $(2,3)$ and $(3,4)$ are both excellent, since $\frac{35}{7} = 5$ and $\frac{91}{13} = 7$. The leads us to conjecture that taking $b = a + 1$ always gives an excellent pair. We can prove this by computing

$$\frac{a^3 + (a+1)^3}{a(a+1)+1} = \frac{2a^3 + 3a^2 + 3a + 1}{a^2 + a + 1} = \frac{(2a+1)(a^2+a+1)}{a^2+a+1} = 2a + 1,$$

which is clearly a positive integer.

4. Using finite differences leads to the following rows of numbers.

$x = 1$		2		3		4		5		6
1		2		9		25		53		96
	1		7		16		28		43	
		6		9		12		15		
			3		3		3			

All subsequent rows will contain 0's. Extending the differences back one term in each row gives values of 3, -2, 3, 3 along the diagonal beneath $x = 0$, leading to the formula

$$3 - 2x + 3\,\frac{x(x-1)}{2} + 3\,\frac{x(x-1)(x-2)}{6} = \frac{x^3}{2} - \frac{5x}{2} + 3.$$

5. One quickly suspects that taking $b = a^2$ gives an excellent pair. This is easily confirmed, since $(a^3 + (a^2)^3)/(a(a^2)+1) = (a^6 + a^3)/(a^3+1) = a^3$.

Much less obviously, the pairs $(2,10)$, $(3,17)$, $(4,26)$, $(5,37)$, $(6,50)$, $(7,65)$, and $(8,82)$ are also part of an infinite family. Applying the method of finite differences to the values of b leads us to discover the formula $b = a^2 + 2a + 2$. (Or one could notice that each b-value is one more than a perfect square.) We now need to perform long division on the expression

$$\frac{a^3 + (a^2+2a+2)^3}{a(a^2+2a+2)+1} = \frac{a^6 + 6a^5 + 18a^4 + 33a^3 + 36a^2 + 24a + 8}{a^3 + 2a^2 + 2a + 1}.$$

The quotient comes out to exactly $a^3 + 4a^2 + 8a + 8$ with no remainder, so we're done.