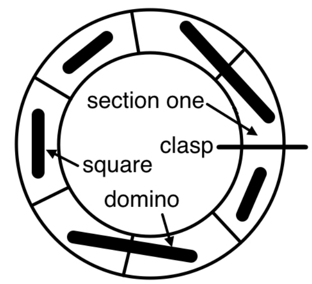
February 2014

The Mandelbrot Team Play

Round Two Test

Time Limit: 60 minutes

Facts: The *Lucas numbers* are $L_1 = 1$, $L_2 = 3$, $L_3 = 4$, $L_4 = 7, \ldots$ in which each term of the sequence from L_3 on is equal to the sum of the previous two terms. It is known that L_n counts the number of ways to *tile a bracelet with clasp* of length n with squares and dominoes. A bracelet of length 7 (having seven sections) is shown along with the tiling *dssds*, where *d* represents 'domino,' *s* stands for 'square,' and we list the tiles in counterclockwise order, beginning with the tile covering section one. We obtain the new tiling *sdssd* by moving all tiles one section counterclockwise. Rotating the tiles clockwise instead gives the new tiling *Dssds*, where we capitalize the first *D* to indicate that the first domino now covers the clasp.



Setup: Rows one through five of the *Lucas triangle* appear at right. (Row zero has deliberately been omitted.) The left and rightmost entries of each row are defined to be 1 and 2, while each interior entry is the sum of the entry above and the entry diagonally up to the left. We label the entries of row n as entry 0 through entry n, and denote the number in row n, entry k by $L(n, k)$. So $L(5, 2) = 14$, for instance. For $n \geq 1$, it is a fact that $L(n, k)$ counts the number of ways to tile a bracelet with clasp of length $n + k$ using k dominoes and $n - k$ squares.

?					
1	2				
1	3	2			
1	4	$\boxed{5}$	2		
1	$\boxed{5}$	9	7	2	
$\boxed{1}$	6	14	16	9	2

Problems

Part i: (4 points) Write down rows six to ten of the Lucas triangle. Also confirm that there are indeed $L(5, 2) = 14$ tilings of a bracelet of length 7 using 2 dominoes and 3 squares.

Part ii: (5 points) Based solely on what $L(n, k)$ counts, explain why we have $L(n, 0) = 1$, $L(n, n) = 2$, $L(n, 1) = n+1$ and $L(n, n-1) = 2n-1$. (Don't use induction or the recurrence.)

Part iii: (4 points) Guess a formula for $L(n, n-2)$ when $n \geq 2$. Then prove your conjecture is correct based only on what $L(n, k)$ counts. (Don't use induction or the recurrence.)

Part iv: (5 points) By whatever means, prove that adding the entries within the Lucas triangle along a line of slope 1 (such as the boxed numbers above) gives a Lucas number. Given this result, what is the proper way to define $L(0, 0)$ at the top of the Lucas triangle?

Part v: (5 points) For $n \geq 1$, what is the sum of the entries of row n? Prove your conjecture is correct based only on what $L(n, k)$ counts. (Don't use induction or the recurrence.)

Part vi: (5 points) For a prime p, prove that all entries in the Lucas triangle along the line of slope 1 through $L(p, 0)$ are divisible by p, except for $L(p, 0)$. (See boxes above for $p = 5$.)