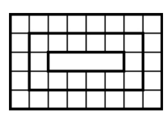
Answer Key

1. $1\frac{1}{2}$	4. $\sqrt{5}-1$
2. $\frac{1}{2}$	5. 9
3. 9696	6. 44
	7. 7

1. Since three fire trucks can pump 1200 gallons in half a minute, each of them is able to pump 400 gallons in half a minute, thus five fire trucks can pump a total of 2000 gallons in half a minute. It is now clear that it will take three times as much time to pump 6000 gallons, which comes to $\mathbf{1\frac{1}{2}}$ minutes.

2. There is no systematic way for solving an equation such as this in which the variable appears as an exponent and as a linear term, so we find solutions by inspection. (In other words, we guess.) It doesn't take long to find that $x=1$ satisfies the equation. Clearly $x=2$ is too large, while $x=0$ is too small, so we next try $x=\frac{1}{2}$. This also works, since $9^{1/2}=3$. Hence $x=\mathbf{\frac{1}{2}}$ is the smaller of the two answers.

3. The numbers of squares in the rectangular tracks form an arithmetic progression, so it is possible to compute the total by summing these terms using standard techniques. However, notice that each track fits neatly within the next larger one. Nesting them all within one another gives a solid 97×100 rectangle, with a small 1×4 rectangle missing in the middle. Therefore the total number of squares is equal to $(97)(100)-(1)(4)=\mathbf{9696}$.



4. Translating the graph of $y=x^2$ by t units in the x-direction and $t-1$ units in the y-direction will shift the vertex from $(0,0)$ to $(t,t-1)$ so that it lies on the line $y=x-1$. The new equation of the graph will be $y=(x-t)^2+(t-1)$. Plugging in $y=0$ gives the y-intercept of t^2+t-1. In order for this to also be an x-intercept, we should obtain $y=0$ when substituting $x=t^2+t-1$. In other words, we must have

$$((t^2+t-1)-t)^2+(t-1)=0 \qquad \Longrightarrow \qquad (t^2-1)^2+(t-1)=0.$$

Factoring out $(t-1)$ leads to $t(t-1)(t^2+t-1)=0$, so $t=0$, 1, or $\frac{1}{2}(-1\pm\sqrt{5})$. These give x-intercepts of -1, 1, 0 and 0, respectively. The average of the two x-intercepts is t, the x-coordinate of the vertex. Using this fact we quickly find the other x-intercept in each case to be 1, 1, $-1-\sqrt{5}$ and $\mathbf{-1+\sqrt{5}}$, the latter of which is the largest.

5. It will be convenient to work with congruences mod 41. To begin, notice that mod 41 we have $40\equiv(-1)$, $39^2\equiv(-2)^2$, $38^3\equiv(-3)^3$, all the way up to $21^{20}\equiv(-20)^{20}$ mod 41. Combining like terms and all the negatives yields

$$\begin{aligned} 1^{40}2^{39}\cdots 39^2 40^1 &\equiv 1^{40}(-1)2^{39}(-2)^2\cdots 20^{21}(-20)^{20} \\ &\equiv 1^{41}2^{41}\cdots 20^{41}(-1)^{1+2+\cdots+20}. \end{aligned}$$

But $1+2+\cdots+20=210$ is even, so all the negatives cancel out. Furthermore, we know that $1^{41}\equiv 1$, $2^{41}\equiv 2$, …, $20^{41}\equiv 20$ by Fermat's Little Theorem. Therefore the remainder we are seeking is the same as $(1)(2)(3)\cdots(19)(20)$, reduced mod 41. This can be done by hand without much trouble by strategically combining certain pairs of terms. For instance, $(3)(14)\equiv 1$, $(5)(8)\equiv -1$, and so on. The reader is invited to finish reducing this product; if done correctly the result will be **9**.

6. Label $BC=a$, $AC=b$ and $AB=c$ as customary. The length of the median from C to $\overline{AB}$ is given by $\frac{1}{2}\sqrt{2a^2+2b^2-c^2}$; this is a special case of Stewart's Theorem and a standard fact in its own right. (The reader is invited to research these formulas if they are unfamiliar.) Since G is located $\frac{2}{3}$ of the way along the median, we have $CG=\frac{1}{3}\sqrt{2a^2+2b^2-c^2}$. Of course $CM=\frac{1}{2}a$, so squaring the equality $CG=CM$ yields

$$\frac{1}{9}(2a^2+2b^2-c^2)=\frac{1}{4}a^2,$$

which reduces to $8b^2=a^2+4c^2$. But a, b and c are positive integers, hence a must be even, say $a=2k$. Substituting $a=2k$, $b=13$, and dividing by 4 gives $338 = c^2 + k^2$. There are three possible solutions: either $c = k = 13$, or $c = 7$ and $k = 17$, or $c = 17$ and $k = 7$. The first two options don't lead to actual triangles (they don't satisfy the triangle inequality), so we must have $a = 14$, $b = 13$, and $c = 17$ which yields a triangle with a perimeter of **44**.

7. Let E be the expected number of successful turns for the whole game. If the first number falls between 0 and 1, then the second number will always be larger than the first, and we will be back in the same position as when the game began. On the other hand, if the first number is between 1 and 2 then the situation is more complicated. Let F be the expected number of further successful turns in this case. So far we have

$$E = \tfrac{1}{2}(1 + E) + \tfrac{1}{2}(1 + F).$$

Let us return to the scenario in which the first number is between 1 and 2. If the second number is also between 1 and 2 then half the time the game is over; the other half of the time the third number is automatically larger and it will be as if we just started the game. On the other hand, if the second number is between 2 and 3 then it will definitely be larger than the first number, but we will be back in the same situation as at the start of this paragraph. To summarize,

$$F = \tfrac{1}{2}\left(0 + \tfrac{1}{2}(E + 1)\right) + \tfrac{1}{2}(1 + F).$$

Solving these equations gives $E = \mathbf{7}$.

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