Answer Key	
1. 4	4. 19
2. 4×10^{29}	5. 3410
3. 18	6. 36
	7. $\frac{1}{2}(3-\sqrt{5})$

1. There are exactly four such lines. A line through Q parallel to any of the other three lines will intersect the rest of the diagram in exactly two other points. And drawing the line from Q through the point common to all three given lines results in only one point of intersection with the rest of the diagram. Therefore the total is **4** lines.

2. We compute the ratio as follows:

$$\frac{2 \times 10^{50}}{5 \times 10^{20}} = \frac{2}{5} \times 10^{30} = 0.4 \times 10^{30} = \mathbf{4} \times \mathbf{10^{29}}.$$

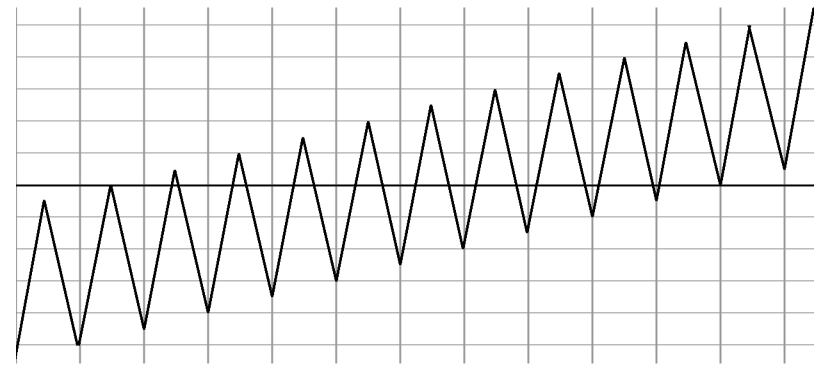
(We "borrow" a 10 from 10^{30} to write the answer in the desired form.)

3. Let t_1, t_2 and t_3 represent Sutton's test scores in order from least to greatest. Then the difference A between his higher two test scores is $A = t_3 - t_2$, and similarly $B = t_2 - t_1$. We wish to compute $A - B$, or $(t_3 - t_2) - (t_2 - t_1) = t_3 - 2t_2 + t_1$. We are told that his average $\frac{1}{3}(t_1 + t_2 + t_3)$ is 6 more than his middle score t_2. Thus

$$\frac{t_1 + t_2 + t_3}{3} = t_2 + 6 \qquad \Longrightarrow \qquad t_1 + t_2 + t_3 = 3t_2 + 18.$$

Subtracting $3t_2$ from both sides yields $t_3 - 2t_2 + t_1 = \mathbf{18}$.

4. Perhaps the most effective approach to this problem is to draw an accurate picture of the graph in the vicinity of $y = 2013$, shown to the right above. (The grid lines are spaced every two units, and the graph is not drawn to scale to better fit on the page.) The heavy horizontal line is the graph of $y = 2013$, while the black zig-zag is the graph of our function. Each intersection between these two graphs represents a solution to $f(x) = 2013$. A careful count reveals **19** such points.



5. Suppose the length and width of our rectangle are x and y. Then there are $(x-1)$ vertical and $(y-1)$ horizontal grid lines within the rectangle, for a total of $(x-1)(y-1)$ grid points inside the rectangle. Furthermore, each of the $(x-1)$ vertical grid lines is split into y unit segments, while each of the $(y-1)$ horizontal grid lines is split into x unit segments. Hence the total score of such a rectangle would be

$$7(x-1)(y-1) - 3y(x-1) - 3x(y-1) = xy - 4x - 4y + 7.$$

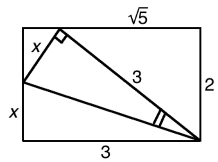
To obtain a total score of 2013 we must have

$$xy - 4x - 4y + 7 = 2013 \qquad \Longrightarrow \qquad (x-4)(y-4) = 2022.$$

But $2022 = 2 \cdot 3 \cdot 337$, and we must have $x, y \geq 8$, which forces us to use $x = 10$, $y = 341$ or vice-versa. Either way the area is $(10)(341) = \mathbf{3410}$.

6. Imagine making a list of the divisors of each number from 1 to 45, with each set of divisors on a separate line. Any number 23 or greater will only appear once, so such a number is a good candidate for the least likely result. Also note that the more divisors a number has, the less likely that any particular divisor of that number will be chosen. A bit of legwork reveals that the positive integer from 1 to 45 with the most divisors is 36, with nine divisors. Therefore **36** is the least likely result: it appears with a probability of only $\frac{1}{45} \cdot \frac{1}{9}$.

7. In diagram below let x represent the length along the left-hand edge that is folded upward. Since the folded triangular portion is congruent to the region it used to occupy in the lower left part of the rectangle, we label the other lengths with 3 and x, as shown. By the Pythagorean Theorem we may next deduce the length along the top to be $\sqrt{5}$. The stage is now set to solve for x via the right triangle in the upper left corner. By subtracting we find that its legs have length $(2-x)$ and $(3-\sqrt{5})$, while the hypotenuse is x. Therefore



$$(2-x)^2 + (3-\sqrt{5})^2 = x^2.$$

Expanding the squares and simplifying yields

$$(4-4x+x^2)+(9-6\sqrt{5}+5)=x^2 \qquad \Longrightarrow \qquad 18-6\sqrt{5}=4x.$$

Hence $x = \frac{1}{2}(9-3\sqrt{5})$. We are now able to compute $\tan\alpha$ as

$$\tan\alpha = \frac{x}{3} = \mathbf{\frac{3-\sqrt{5}}{2}}.$$

In case you're interested, this happens to be exactly the same as $1/\varphi^2$, where φ is the golden ratio.

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⋆ Regional Level ⋆

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