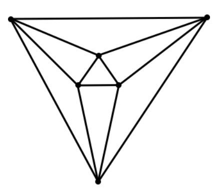
Answer Key	
1. 2314	4. 7/4
2. 8	5. 59
3. 18	6. 2380
	7. $1011 + 91i$

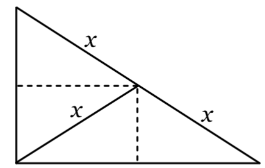
1. Suppose that $a > b$. This means that $a < c$, since exactly one of $a > b$ or $a > c$ is true. So far we have $b < a < c$, thus $b < c$. Condition (ii) now implies $b > d$, giving $d < b < a < c$ overall. But this violates condition (iii), so our original assumption that $a > b$ must be mistaken. Starting with $a < b$ instead and employing the same logic, we arrive at $c < a < b < d$ overall. Hence $c = 1$, $a = 2$, $b = 3$, $d = 4$ and our four-digit number $1000a + 100b + 10c + d$ must be **2314**.



2. Upon drawing a diagram we instinctively lean towards configurations with as few intersections among the segments as possible, since these create more regions. It is actually possible to obtain no intersections, as illustrated here, giving eight regions. In fact, Euler's formula ensures that every such diagram will involve precise eight regions. Each of our six points is the endpoint of four segments, for a total of 24 endpoints, hence there are 12 segments. According to Euler, $V - E + F = 2$ for such configurations, where F is the number "faces" (regions). In our case $V = 6$ and $E = 12$, so we must have $F = 8$. Therefore **8** regions is the best possible.

3. Some experimentation reveals that for any positive integer n one of $4 + n$, $6 + n$, $8 + n$ is composite. In retrospect, this makes sense because exactly one of them will be a multiple of 3. (Because $8 + n$ is a multiple of 3 exactly when $5 + n$ is.) Since 4, 6, 8 are the first three composite numbers, $f(n)$ will always be one of these three numbers. In fact, $f(2) = 4$, $f(3) = 6$ and $f(1) = 8$, so all three values occur, meaning that the desired sum is $4 + 6 + 8 = \mathbf{18}$.

4. It is possible to bash this algebraically, but there is a more elegant and illuminating geometric approach. Place the two isosceles triangles next to one another as shown. We claim that they must fit together to form a large right triangle. To see why, split the isosceles triangles along the dotted lines into two right triangles. Since all four small right triangles have the same area, we can rearrange the pieces of one isosceles triangle to form the other, which explains the claim above. By the Pythagorean theorem, we now deduce



$$(\sqrt{6})^2 + (\tfrac{5}{2})^2 = (2x)^2 \quad \Longrightarrow \quad 6 + \frac{25}{4} = \frac{49}{4} = 4x^2,$$

hence $x^2 = \frac{49}{16}$, which gives $x = \mathbf{\frac{7}{4}}$.

5. It is tempting to multiply out, but instead we employ the formula $1 + 2 + \cdots + n = \frac{1}{2}n(n + 1)$ to rewrite the expression as

$$\tfrac{1}{2}(1)(2) \cdot \tfrac{1}{2}(2)(3) \cdot \tfrac{1}{2}(3)(4) \cdots \tfrac{1}{2}(59)(60).$$

Combining corresponding terms, we discover that we are dealing with two factorials divided by a large power of 2; namely

$$\frac{(59!)(60!)}{2^{59}} = \frac{2(59!)(60!)}{2^{60}},$$

where we have included the extra 2 to capitalize on the fact that $2^{60} \equiv 1 \bmod 61$ by Fermat's Little Theorem. Moreover, by Wilson's Theorem $60! \equiv -1$ and $59! \equiv 1 \bmod 61$, hence our entire expression reduces to just $-2 \equiv 59 \bmod 61$. Therefore the desired remainder is **59**.

6. It is well-known that the number of paths from the bottom left to top right corner of the grid is $\binom{8}{4} = 70$, since we must choose four of the eight unit segments making up the path to be vertical, with the other four being horizontal. However, this is not the final answer, since as Fran moves the chip along a particular path she can advance it either one or two units at a turn. In other words, there are multiple ways to trace out a particular path using the allowable moves.

Looking carefully, we see that at any point she always has precisely these two options—to advance the chip one or two units along the path. Hence for any given path we must count the number of ways to move eight units forward if at each turn we may advance either one or two units. This is also a well-known problem; the reader may confirm that there are F_{n+1} ways to advance n units in this manner. Therefore in total there are $70F_9 = 70(34) = \mathbf{2380}$ ways for Fran to move the chip from the bottom left to the top right corner.

7. Although not immediately recognizable, the numbers in the denominator are all of the form n^2+1, for n ranging from $n = 11$ to $n = 100$. Complex numbers are present, so we opt to factor this expression as $(n+i)(n-i)$. The factors in the numerator are even less recognizable, but the last term tips us off to the fact that these terms have the form $(n^2+n+1+i)$, from $n = 10$ to $n = 100$. (In particular, note there is one more factor in the numerator!) One might hope that these terms also factor; happily we discover that

$$(n^2+n+1+i) = (n+i)(n+1-i).$$

Therefore the entire expression can be rewritten as

$$\frac{(10+i)(11-i)(11+i)(12-i)(12+i)(13-i)\cdots(100+i)(101-i)}{(11+i)(11-i)(12+i)(12-i)\cdots(100+i)(100-i)},$$

where the offset helps us to remember the extra factors in the numerator. Cancelling everything in sight simplifies the expression down to just

$$(10+i)(101-i) = \mathbf{1011} + \mathbf{91}i.$$

Problem Credits

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⋆ National Level ⋆

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