Answer Key

1. 15
2. 165
3. $15\pi - 18\sqrt{3}$
4. 26
5. $2\pi^2$
6. 21
7. 304

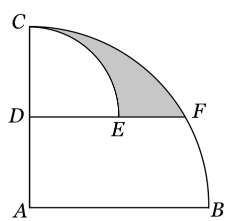
1. Some experimentation suggests that the optimal arrangement of digits is akin to the diagram at left, giving a minimal largest product of 15.

4	1	6
3	5	2

To see that it's not possible to go lower, note that the 6 and 5 cannot both be adjacent to only the 1 and 2. Hence at least one of them is next to a 3 or higher, meaning that some product will be at least 15 or higher; thus **15** must be the answer.

2. The key to any solution, algebraic or otherwise, is to realize that in order for the concentration of salt to increase by a factor of three, there must be only one-third as much water in the cup after the evaporation. Armed with this insight, we now see that if 50 grams of water evaporated, this must represent two-thirds of the original amount of water, hence we began with 75 grams of water. Subtracting this out, along with the 10 grams of salt, leaves $250 - 75 - 10 = \mathbf{165}$ grams for the cup.

3. Since the radius of the circle is 12, we know $AF = 12$, and since D is a midpoint, we also have $AD = 6$. Hence $\triangle ADF$ is a $30°-60°-90°$ triangle, with $DF = 6\sqrt{3}$ and $area(ADF) = 18\sqrt{3}$. This means that $m\angle BAF = 30°$, so the area of sector BAF is a third of the entire quarter circle, giving an area of $\frac{1}{3}(\frac{1}{4}\pi 12^2) = 12\pi$. Finally, the area of quarter circle CDE is $\frac{1}{4}(\pi\, 6^2) = 9\pi$. Subtracting off all three of these areas from the original quarter circle CAB yields our answer of



$$\tfrac{1}{4}(\pi 12^2) - 9\pi - 12\pi - 18\sqrt{3} = \mathbf{15\pi - 18\sqrt{3}}.$$

4. The chart below tabulates all permissible pairs a^b and c^d such that $a^b c^d = 3^6 5^8$. Its organization helps to make it clear that all possibilities are included. (In each column, let a^b be the exponential with the smaller base, to satisfy $a < c$.) There are a total of **26** ways in all.

$(3^1)^2$	$(3^1)^2$	$(3^1)^4$	$(3^1)^6$	$(3^1)^6$	$(3^1)^6$	$(3^2)^2$
$(3^2 5^4)^2$	$(3^1 5^2)^4$	$(3^1 5^4)^2$	$(5^1)^8$	$(5^2)^4$	$(5^4)^2$	$(3^1 5^4)^2$

$(3^2)^3$	$(3^2)^3$	$(3^2)^3$	$(3^3)^2$	$(3^3)^2$	$(3^3)^2$	$(3^1 5^1)^2$	$(3^1 5^1)^4$
$(5^1)^8$	$(5^2)^4$	$(5^4)^2$	$(5^1)^8$	$(5^2)^4$	$(5^4)^2$	$(3^2 5^3)^2$	$(3^1 5^2)^2$

$(3^1 5^1)^6$	$(3^2 5^1)^2$	$(3^2 5^1)^3$	$(3^3 5^1)^2$	$(3^3 5^1)^2$	$(3^3 5^1)^2$
$(5^1)^2$	$(3^1 5^3)^2$	$(5^1)^5$	$(5^1)^6$	$(5^2)^3$	$(5^3)^2$

$(3^2 5^2)^2$	$(3^2 5^2)^3$	$(3^3 5^2)^2$	$(3^3 5^2)^2$	$(3^3 5^3)^2$
$(3^1 5^2)^2$	$(5^1)^2$	$(5^2)^2$	$(5^1)^4$	$(5^1)^2$

5. What is not apparent from the question is that the squares are tilted 45° from the horizontal. However, this becomes clear upon realizing that taking $y = \frac{\pi}{2} + x$ gives

$$\cos y = \cos(\tfrac{\pi}{2} + x) = -\sin x,$$

so the line $y = \frac{\pi}{2} + x$ is part of our graph. The most efficient way to finish is to find and plot all the x- and y-intercepts. These occur where $\sin x + \cos 0 = 0$ and $\sin 0 + \cos y = 0$, so at the points $(-\frac{5\pi}{2}, 0)$, $(-\frac{\pi}{2}, 0)$, $(\frac{3\pi}{2}, 0)$ and $(0, -\frac{\pi}{2})$, $(0, \frac{\pi}{2})$, $(0, \frac{3\pi}{2})$, for instance. Sketching in the diagonal lines through these points reveals that each square has a diagonal of length 2π, and hence an area of $\frac{1}{2}(2\pi)^2 = \mathbf{2\pi^2}$.

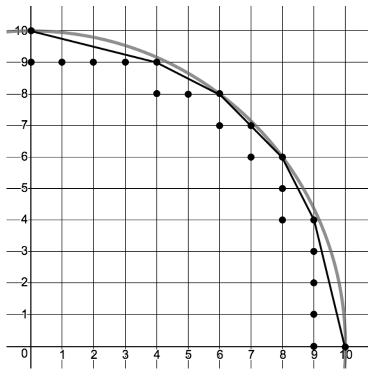
6. Let the number of blocks of each color be a, b, c, d, where b represents the number of blue blocks. We are given that

$$ab + ac + ad + bc + bd + cd = 157, \qquad ac + ad + cd = 119.$$

Subtracting gives $b(a + c + d) = 38$. Since there is at least one block of each color, $a + c + d \geq 3$, meaning that either $b = 1$ or $b = 2$. But $b = 1$

leads to $a+c+d=38$ and $ac+ad+cd=119$, which has no integer solutions. (Try $a=1$, 2 and 3 to see why.) We are left with $b=2$ and $a+c+d=19$, giving an overall sum of **21** blocks, for example with $a=5$, $c=7$, $d=7$.

7. A quarter of the polygon, along with a few of those 317 points near the circle, is shown below.



It is now evident that there are 24 points along the boundary of this polygon; four on the coordinate axes and five in each quadrant. This leaves $317-24=293$ points in its interior. Hence by Pick's Theorem the area of this polygon is

$$\tfrac{1}{2}B+I-1=\tfrac{1}{2}(24)+293-1=\mathbf{304}.$$

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Problem Credits

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⋆ Regional Level ⋆

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