February 2013

The Mandelbrot Team Play

Round Two Test

Time Limit: 60 minutes

Facts: Given real numbers a and b, the value of $\min(a, b)$ is equal to the smaller of a and b. If a and b happen to be equal, then $\min(a, b)$ equals this common value. Thus $\min(\pi, \sqrt{10}) = \pi$, while $\min(\frac{8}{5}, 1.6) = 1.6$.

Each term in an *arithmetic sequence* differs from the previous one by a fixed amount, as in the sequence 4, 7, 10, 13, ..., 100. To find their sum, count how many terms there are, compute the average value of the terms, and multiply these quantities. In our case there are $\frac{1}{3}(100 - 4) + 1 = 33$ terms, the average is $\frac{1}{2}(4 + 100) = 52$, for a sum of $33 \cdot 52 = 1716$.

Setup: Let $a_1 = x$ and $b_1 = y$ be positive real numbers. Obtain a second pair of numbers by setting $a_2 = \min(a_1, b_1)$ and $b_2 = |a_1 - b_1|$. Then iterate this process by letting $a_3 = \min(a_2, b_2)$ and $b_3 = |a_2 - b_2|$, then $a_4 = \min(a_3, b_3)$ and $b_4 = |a_3 - b_3|$, and so forth. Finally, define

$$g(x, y) = a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4 + \cdots.$$

In other words, given x and y, use these as the first pair of values for the above process, generate the subsequent pairs of numbers, then add together the products of all the pairs. For example, one finds that $g(2, 5) = 19$. Our goal will be to determine the value of $g(\pi, 1)$.

Problems

Part i: (4 points) Compute the values of $g(3, 7)$, $g(10, 17)$ and $g(5, 100)$.

Part ii: (4 points) Show that for any positive integer k we have $g(2, 2k) = 2k^2 + 2k$. Then find a formula for $g(2, 2k + 1)$ in terms of k.

Part iii: (5 points) Starting with $a_1 = x$ and $b_1 = y$, then defining a_2, b_2, ... as described above, show that $(a_2 + a_3 + \cdots + a_n) + (a_n + b_n) = x + y$ for any $n \geq 2$. Use this fact to explain why $a_2 + a_3 + a_4 + \cdots \leq x + y$.

Part iv: (5 points) Suppose that $x, y < \epsilon$ for some small positive real number $0 < \epsilon < 1$. Use the result of the previous part to prove that $g(x, y) < 3\epsilon$.

Part v: (5 points) Demonstrate that $g(a_n, b_n) - \frac{1}{2}(a_n^2 + a_n b_n + b_n^2)$ is an invariant of this process, meaning that the value of this expression is the same for any $n \geq 1$.

Part vi: (5 points) Put everything together to find, with proof, the value of $g(\pi, 1)$.