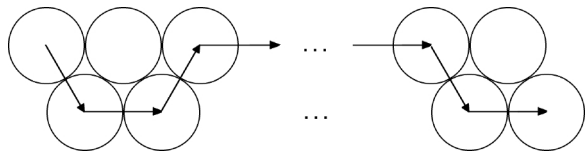
Team Play Topics

Round One

The first section of the Round One Mandelbrot Team Play is reproduced below. A list of topics and practice problems are also provided to aid in preparation. Note that these problems are not meant to serve as a precise indicator of the problems that will appear on the contest. However, students who understand how to solve them should be able to make significantly more progress than they might have otherwise. So work hard on the problems, and good luck on the Team Play!

Facts: Consider a row of n adjacent circles with a second row of n circles beneath and slightly to the right, just touching the first row, as shown below. A path through these circles consists of a sequence of steps from the upper left to lower right circle, always moving directly to the right, up/right, or down/right. One such path is illustrated in the diagram. There are a total of F_{2n} paths through the circles, where F_{2n} is a Fibonacci number, defined by $F_1 = 1$, $F_2 = 1$, and $F_{k+1} = F_k + F_{k-1}$ for $k \geq 2$.



Topics: Fibonacci numbers, counting techniques, partitions, bijections.

Practice Problems

1. Create two rows of three circles each as described above and show that there are a total of eight paths from the upper left to lower right circle. Which Fibonacci number is 8?

2. It turns out that in the previous problem there are five paths from the upper left to upper right circle. Now extend each row to four circles. Without actually making a list, show that there are 13 paths to the new upper right circle. How many paths are there to the new lower right circle?

3. Demonstrate that the claim made in the Facts section above is valid.

4. Given two rows of ten circles, how many paths pass through the fourth circle in the top row? How many paths do *not* pass through this circle? (Write your answers as a product of two Fibonacci numbers.) What fact involving Fibonacci numbers may be deduced?

5. Generalize the observation made in the previous problem, and prove your identity using paths.

Hints and answers on the next page. $\Longrightarrow$

Team Play Topics

Hints and Answers

1. We leave it to the reader to find the eight paths. (Three start off to the right, the other five begin down/right.) Note that $8 = F_6$.

2. Any path ending at the upper right circle must reach that circle by moving right or up/right at the last step. But there are 5 paths to the third circle in the top row and 8 paths to the third circle in the bottom row, for a total of 13 paths. The same reasoning reveals that there are $8 + 13 = 21$ paths to the bottom right circle with two rows of four circles.

3. One can prove this claim by induction. We claim that there in two rows of n circles there are F_{2n-1} paths to the upper right circle and F_{2n} paths to the lower right circle. The reader should verify this claim for $n = 1$ and $n = 2$, establishing the base cases. Just as in the previous problem, to reach the upper right circle we must move either right or up/right at the last step. By the inductive hypothesis there are F_{2n-3} and F_{2n-2} ways to make it this far. Adding these gives F_{2n-1}, as desired. The argument for the lower right circle is similar, giving $F_{2n-2} + F_{2n-1} = F_{2n}$.

4. There are F_7 ways to reach the fourth circle in the top row, and F_{14} ways to move from there to the lower right circle, for a total of F_7F_{14} such paths. To avoid the fourth circle in the top row we must first move to the third circle in the bottom row (F_6 ways), then move right, then finish from there (F_{13} ways), for a total of F_6F_{13} such paths. We deduce that $F_6F_{13} + F_7F_{14} = F_{20}$, the total number of paths of any sort.

5. In general we conjecture that $F_aF_b + F_{a+1}F_{b+1} = F_{a+b+1}$ for and positive integers a and b. This may be proven by looking at two rows having a total of $a + b + 1$ circles (if this quantity is odd then the top row will be one circle longer than the bottom row) and considering paths which do or do not pass through the $(a + 1)^{\text{st}}$ circle.