March 2011

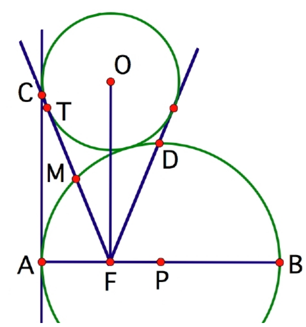
# The Mandelbrot Team Play

## Round Three Test

*Time Limit: 60 minutes*

**Facts:** The main results outlined on this Team Play rely on a variety of tools from geometry and trigonometry. These include the facts that an inscribed angle is equal to half its intercepted arc, that a quadrilateral $PQRS$ is cyclic if and only if $\angle PQS \cong \angle PRS$, and the Pythagorean Theorem. Trigonometric facts include right triangle trig, the Law of Cosines, and elementary identities such as $\sin(x + 90°) = \cos x$ or $\cos(x + 90°) = -\sin x$.

**Setup:** In the diagram at right $\overline{AB}$ is a diameter of the larger circle with center $P$ and the vertical line $l$ is tangent to this circle at $A$. We now draw a smaller circle with center $O$ tangent to both the larger circle and to $l$. Let $F$ be the foot of the perpendicular from $O$ to $\overline{AB}$, and draw tangents from $F$ to the smaller circle. Let the left-hand tangent touch the circle at $T$ and intersect $l$ at $C$, while the right-hand tangent meets the larger circle at $D$.




**Problems**

**Part i: (4 points)** Let $R$ and $r$ be the radii of the larger and smaller circle, respectively. Compute lengths $OP$, $FP$ and $OF$ in terms of $R$ and $r$.

**Part ii: (5 points)** Prove that triangles $\triangle CAF$ and $\triangle FTO$ are congruent.

**Part iii: (4 points)** Explain why $\cos(m\angle CFP) = -\frac{1}{2}\sqrt{\frac{r}{R}}$.

**Part iv: (5 points)** Let $M$ be the midpoint of $\overline{CF}$. Show that $PM = PA$. Be careful not to assume that $M$ lies on the larger circle just because it appears that way in the diagram.

**Part v: (5 points)** Prove that $\angle ADF \cong \angle ACF$.

**Part vi: (5 points)** Finally, prove that $\triangle CDF$ is a right triangle and that $\triangle BDF$ is an isosceles triangle.