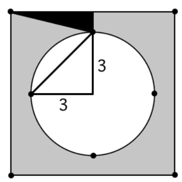
Answer Key		4.	$6,\ 5,\ 9$
1.	$1\frac{2}{13}$	5.	$1620/\pi$
2.	$3\sqrt{2}$	6.	4
3.	176	7.	768

1. Converting each mixed fraction to an improper fraction, then flipping each fraction in the denominator, leads to

$$\left(\frac{7}{2}\cdot\frac{13}{3}\cdot\frac{21}{4}\cdots\frac{73}{8}\right)\left(\frac{4}{13}\cdot\frac{5}{21}\cdot\frac{6}{31}\cdots\frac{10}{91}\right)=\frac{7\cdot 9\cdot 10}{2\cdot 3\cdot 91},$$

after a delightful amount of cancellation. This reduces to $\mathbf{\frac{15}{13}}$, or $\mathbf{1\frac{2}{13}}$, since it feels appropriate to write our answer as a mixed fraction.



2. With a bit of trial and error we discover the unique arrangement of eight points within region $\mathcal{R}$ for which each pair of points are no closer that $\sqrt{17}$ of one another, shown at right, using the four corners of the square and the four "compass points" of the circle. Indeed, the shaded right triangle has legs of length 1 and 4, so the hypotenuse has length $\sqrt{17}$. By inspecting the diagram, we see that the next shortest distance will occur using two adjacent points on the circle. By the Pythagorean Theorem, this distance is $\sqrt{3^2+3^2}=\sqrt{\mathbf{18}}=\mathbf{3\sqrt{2}}$.

3. There are 7 choices for which letter to remove, then 26 possible letters that could replace it, so it would appear we have $7\cdot 26=182$ sets of letters that could result. The catch is that seven of these sets are actually identical: we could remove F then replace it with F, or remove R and replace it with R, and so forth, each time getting back to 'FRIENDS' where we began. Since these seven sets only represent one distinct set of letters, our total should be reduced to $182-6=\mathbf{176}$.

4. Notice that the bottom pair of equations cannot both be true, since if they were we would have

$$3x+2z+12=6y,\quad z+6=3y\quad\Longrightarrow\quad 3x=0,$$

by subtracting twice the second equation from the first. This would give $x=0$, when all values should be nonzero. Similarly, the left pair of equations cannot both be true, since that would imply $z=0$. We deduce that the bottom left equation must be false, for if it were true then two other equations would be false. It is now routine to solve the remaining three equations: one efficient method is to substitute $x=2y-4$ and $z=3y-6$ into the top right equation. This leads to $y=5$, from which we get $x=6$ and $z=9$. (Written in order we have $\mathbf{6,5,9}$.)

5. Illustrating this solution proved to be an insurmountable challenge, so the reader will have to use their imagination. Using standard facts relating the equation of an ellipse to its shape, we conclude that the graph of $\frac{x^2}{100}+\frac{y^2}{19}=1$ will be an ellipse having a horizontal major axis with length $2\sqrt{100}=20$ and vertical minor axis with length $2\sqrt{19}$. The foci are at $(\pm c,0)$, where $c^2=100-19=81$, and hence $c=9$. In other words, the foci are a distance of 18 apart, so when the paper is rolled up into a cylinder the circular cross-section will have circumference 18. But this means its radius r satisfies $2\pi r=18$, so $r=9/\pi$. Therefore its area equals $\pi r^2=81/\pi$, which results in a volume of $20\cdot(81/\pi)=\mathbf{1620}/\pi$.

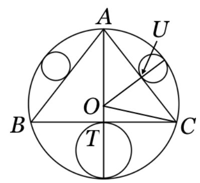
6. There are various ways to organize a search, for instance by looking for degree 0 polynomials, then degree 1, and so forth. Here is an illuminating alternate approach. We first note that $P(x)=17$ is the unique degree 0 polynomial. Next observe that any such polynomial with degree 1 or more corresponds to a base b representation of 17 in which the sum of the base b digits is equal to b. For instance, 17 written in base 5 is 32_5, and $3+2=5$. This corresponds to the polynomial $P(x)=3x+2$, for which $P(1)=5$ and hence

$$P(P(1))=P(5)=3(5)+2=17.$$

Therefore we need only tabulate the base b representations of 17.

base	2	3	4	5	6	7	8	9	10	$\cdots$	17
digits	10001	122	101	32	25	23	21	18	17	$\cdots$	10

We find three solutions: $P(x) = x^4+1$, $P(x) = 3x+2$, and $P(x) = x+8$, corresponding to $b = 2$, $b = 5$, and $b = 9$. Along with $P(x) = 17$ this gives a total of **4** such polynomials.



7. Let the circumcircle have center O and radius r. Drawing the segments from O shown in the diagram results in a plethora of right triangles, enough to enable us to solve for r. By symmetry we have $\overline{AT} \perp \overline{BC}$, $\overline{OU} \perp \overline{AC}$, and $AU = UC$. We will ultimately invoke the Pythagorean Theorem on $\triangle ATC$, but first note that $\triangle OTC$ gives

$$(TC)^2 = r^2 - (r-18)^2 = 36r - 324,$$

using the fact that $OT = r - 18$ due to the size of the smaller inscribed circle. We will also need to compute

$$(AC)^2 = 4(AU)^2 = 4\left(r^2 - (r-10)^2\right) = 80r - 400,$$

via Pythagorean on $\triangle AUO$. Finally, we return to $\triangle ATC$ to obtain

$$(AT)^2 + (TC)^2 = (AC)^2 \implies (2r-18)^2 + (36r - 324) = 80r - 400,$$

which reduces to $r^2 - 29r + 100 = 0$. This has solutions $r = 4$ or $r = 25$, of which only the latter is large enough. Hence $TC = 24$, $AT = 32$, and the area of $\triangle ABC$ is **768**.

Problem Credits

1. Dr.V
2. Isaac
3. Isaac
4. Josh
5. Oliver
6. Isaac
7. Josh

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