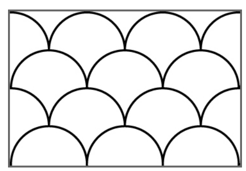
Answer Key	
1. A	4. 502
2. 6π	5. 7/16
3. 2^{2008}	6. $a = 3, b = 10^*$
	7. 46

*See below for an alternate acceptable answer.

1. The three quantities simplify to $4\sqrt{5} + \sqrt{5} = 5\sqrt{5}$, $\frac{21}{4} \cdot \frac{8}{7} = 6$ and $4 - \pi - 7 + \pi = -3$. The first is not an integer, so the answer is **A**.

2. Based on the dimensions of the rectangle, we see that each circular arc in the diagram is part of a circle with diameter 1 cm. There are ten semicircles in the picture, which combine to create five whole circles. There are also four quarter-circles, which form one additional complete circle, for a total of six circles. Each circle has diameter 1cm, and hence circumference π cm, for a total length of $\boldsymbol{6\pi}$ cm.



3. Since $2^{10} = 1024$ the numerator of the fraction equals 2^{1024}. On the other hand, the denominator becomes $(2^2)^{10} = 2^{20}$. Subtracting exponents, the entire fraction simplifies to

$$\left(\frac{2^{1024}}{2^{20}}\right)^2 = (2^{1004})^2 = \mathbf{2^{2008}}.$$

4. We claim that each time Evan makes a cut he introduces four more edges to the overall edge count, regardless of where he chooses to make the cut. This occurs because he splits two of the existing edges in half, so where there used to be two edges there are now four, giving two additional edges. Furthermore, the cut itself creates two new edges, one on each of the resulting pieces. Hence there are four additional edges, as claimed. There are initially 3 edges, so after c cuts there will be $3 + 4c$ edges. We want $3 + 4c = 2011$, which implies that $c = \mathbf{502}$.

5. Points A and B could be located along the top, bottom, left or right edge of the square, leading to a total of sixteen equally likely cases. In five of them A is definitely above B; for instance, when A is on the right edge and B is along the bottom edge. For another five cases B is always above A, and in two cases they are situated at the same height; namely, when A and B are both on the top or bottom edges. This leaves the four cases when A and B are both on a vertical edge of the square. By symmetry, A will be higher half the time, so in effect A has a larger y-coordinate in seven of the sixteen cases, giving a probability of **7/16**.

6. First observe that

$$f(x) = (x^2)^2 - (7x+1)^2 = (x^2 + 7x + 1)(x^2 - 7x - 1),$$

so $f(x)$ has roots $\frac{1}{2}(-7 \pm 3\sqrt{5})$ and $\frac{1}{2}(7 \pm \sqrt{53})$. In order for $f(g(x))$ to be divisible by $x^2 + 9x + 19$, it must be the case that the roots of $f(g(x))$ include the roots of $x^2 + 9x + 19$, which are $\frac{1}{2}(-9 \pm \sqrt{5})$. In other words, plugging $x = \frac{1}{2}(-9 \pm \sqrt{5})$ into $g(x) = ax + b$ must give $\frac{1}{2}(-7 \pm 3\sqrt{5})$ or $\frac{1}{2}(7 \pm \sqrt{53})$. But since a and b are integers it is impossible to obtain $\frac{1}{2}(7 \pm \sqrt{53})$. Furthermore, since a is positive we must have

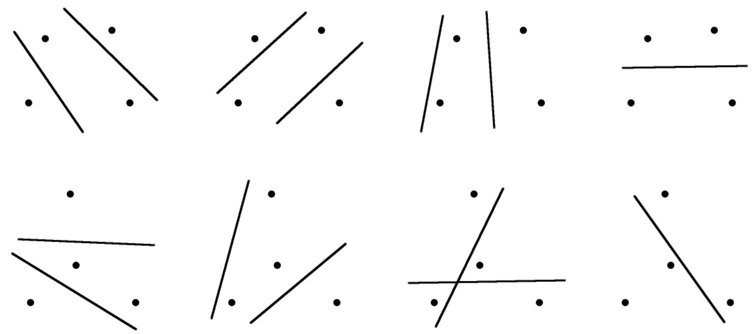
$$a\left(\frac{-9+\sqrt{5}}{2}\right) + b = \frac{-7+3\sqrt{5}}{2},$$

since comparing coefficients of $\sqrt{5}$ indicates we must have $a = \mathbf{3}$. (The other way we would have $a = -3$.) It is routine to solve for b from here, giving $b = \mathbf{10}$. Note that $\boldsymbol{3x + 10}$ is also an acceptable answer.

7. We first transform this problem into a statement concerning points in the plane by employing a stereographic projection through point Z. The points P_1 through P_{10} become ten points Q_1 through Q_{10}, no three of which are collinear. (Since otherwise their preimages, along with Z, would give four points on the same circle.) The planes through Z intersect the sphere in circles. These circles, which are used to split the points into two sets, become lines in the plane under the stereographic projection. So the question now becomes, given ten points in the plane,

no three collinear, how many ways are there to split them into two sets (one of which may be empty) using a line?

For instance, the diagrams below demonstrate that there are exactly seven ways to split four points into two subsets, regardless of the configuration of the points.



It is curious that the number of ways to perform the partition does not depend upon the exact positions of the points. We give an outline for the reason momentarily. For now, computing the number of splits for $n = 1$ to $n = 5$ points gives the first half of the table below.

n	1	2	3	4	5	6	7	8	9	10
splits	1	2	4	7	11	16	22	29	37	46

The values in the bottom row increase by 1, 2, 3 and 4. This pattern persists, leading to **46** ways to divide ten points into two subsets.

Imagine placing a fifth point Q_5 into one of the above diagrams and drawing a line through it. As this line through Q_5 rotates it encounters the other four points one at a time, giving rise to essentially four different ways to split the original points in half. Now translating such a line in either direction off Q_5 gives two ways to split the five points. In other words, we obtain four additional ways to split five points, which is why there are $7 + 4 = 11$ ways listed in the table. The interested reader should fill in the remaining details.

January 2011

⋆ National Level ⋆

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