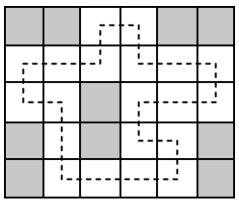
Answer Key	
1. 5	4. 12
2. 10911	5. 3025
3. 7	6. 7
	7. 9/17

1. It is possible to fit **5** dominoes onto a 5×6 board and still have room to draw a connected loop through the remaining squares, as the diagram at left illustrates. To show that it's not possible to squeeze a sixth domino onto the grid, one argues that there is only enough room for four dominoes outside the loop, and then at most one inside the loop, giving our max of five. (Placing two dominoes inside the loop only allows two outside, which is not optimal.)



2. The statement $f(1) = 21$ translates to $a + b + c = 21$, which doesn't narrow things down much. But $f(10) = 201$ is more helpful: this implies that $100a + 10b + c = 201$, which forces $a < 3$ since a, b, c are positive. In fact $a = 2$ is no good either, since this leads to $10b + c = 1$. Hence $a = 1$, so our conditions reduce to $b + c = 20$ and $10b + c = 101$, which gives $b = 9$, $c = 11$. Therefore

$$f(100) = 10000a + 100b + c = \mathbf{10911}.$$

3. For a number to have a base b expansion that begins $3.1\ldots$, it must be the case that this number lies between 3.1 and 3.2 in base b. In other words, the number falls between $3 + \frac{1}{b}$ and $3 + \frac{2}{b}$. So the question boils down to finding for which positive integers b it is the case that

$$3 + \frac{1}{b} \; < \; \pi \; < \; 3 + \frac{2}{b}.$$

Since $3 + \frac{1}{7} \approx 3.142857$ while $\pi \approx 3.1416$, we see that $b = 7$ is too small. However, for all $b \geq 8$ the first inequality is satisfied. On the other side, clearly $b = 14$ works. However we find that $3 + \frac{2}{15} \approx 3.1333$, so the second inequality fails for $b \geq 15$. Hence $8 \leq b \leq 14$, a total of **7** values.

4. This question would appear to be hopelessly time-consuming. In fact, the next primeval decade will be from 2080–2090, but who has the time to check that? However, the alert solver will observe that the only possible years within a decade that can be prime are the ones ending in the digits 1, 3, 7, 9, which is enough information to answer the question. Let n be opening year in the decade, such as $n = 1480$ for the example given in the problem. Then the primes would be $p_1 = n+1$, $p_2 = n+3$, $p_3 = n+7$, and $p_4 = n+9$. Therefore

$$\begin{aligned} p_2p_3 - p_1p_4 &= (n+3)(n+7) - (n+1)(n+9) \\ &= (n^2 + 10n + 21) - (n^2 + 10n + 9) \\ &= \mathbf{12}, \end{aligned}$$

regardless of the decade n that is primeval.

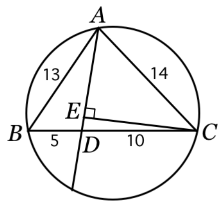
5. To begin, there are a total of $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$ cards in the collection. Let's call this number T. Since each card is equally likely to be chosen, on average we will draw each card once in the course of T trials. Overall we obtain a total value of

$$1(1) + 2(2) + 3(3) + \cdots + n(n) = \tfrac{1}{6}n(n+1)(2n+1),$$

adding together the face values among all T cards drawn and using a well-known formula for the sum of the first n squares. Hence on average the value of a card that is drawn is equal to this total divided by $T = \frac{1}{2}n(n+1)$, which simplifies to $\frac{1}{3}(2n+1)$. We are told this quantity should equal 2017, which gives $n = \mathbf{3025}$.

6. This triangle is probably familiar to most contestants; for instance its area, found via Hero's formula, is 84. Thus $area(ADC) = \frac{2}{3}(84) = 56$, a fact that we will need shortly. Meanwhile, using Stewart's formula (or several applications of the Law of Cosines), we find that

$$15(AD^2) = 10(13^2) + 5(14^2) - 15(5)(10) \quad \Longrightarrow \quad AD = 8\sqrt{2}.$$



But $\overline{CE}$ is the altitude of $\triangle ADC$ corresponding to base $\overline{AD}$, so by computing areas in two different ways we find $\frac{1}{2}(8\sqrt{2})(CE) = 56$, or $CE = 7\sqrt{2}$. Finally, the Pythagorean Theorem on $\triangle CDE$ reveals that $ED = \sqrt{2}$, which leads to

$$\mathit{area}(CDE) = \tfrac{1}{2}(7\sqrt{2})(\sqrt{2}) = \mathbf{7}.$$

7. The entire sum is the product of three geometric series, namely

$$(1 + r + r^2 + \cdots)(1 + s + s^t + \cdots)(1 + t + t^2 + \cdots) = \sum_{k,m,n=0}^{\infty} r^k s^m t^n.$$

Since each of r, s, t are between -1 and 1, the values of these series may be found using the standard geometric series formula, giving

$$\left(\frac{1}{1-r}\right)\left(\frac{1}{1-s}\right)\left(\frac{1}{1-t}\right) = \frac{1}{(1-r)(1-s)(1-t)}.$$

Finally, since r, s, t are the roots of $9x^3 + 9x^2 - 1$, we can factor this cubic as

$$9x^3 + 9x^2 - 1 = 9(x-r)(x-s)(x-t).$$

Plugging in $x = 1$ gives $17 = 9(1-r)(1-s)(1-t)$. Using this in our previous expression leads to the overall answer of $\frac{1}{17/9}$, or $\mathbf{9/17}$.

Problem Credits

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⋆ Regional Level ⋆

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