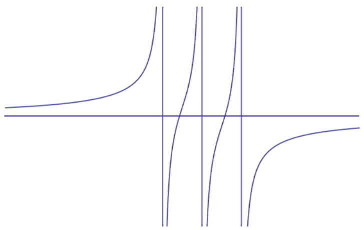
Team Play Solutions

Part i: The equation for the case $n = 3$ is $\frac{1}{1-x} + \frac{1}{2-x} + \frac{1}{3-x} = 0$. The graph will have asymptotes at $x = 1$, $x = 2$, $x = 3$, and the x-axis, as shown at right. (The y-axis is not drawn.) Apparently the solutions a_1 and a_2 are located within the intervals $1 < a_1 < 2$ and $2 < a_2 < 3$.



To solve we multiply through by $(1-x)(2-x)(3-x)$ to obtain

$$(6 - 5x + x^2) + (3 - 4x + x^2) + (2 - 3x + x^2) = 0.$$

This simplifies to $3x^2 - 12x + 11 = 0$, which has solutions $a_1 = 2 - \frac{\sqrt{3}}{3}$ and $a_2 = 2 + \frac{\sqrt{3}}{3}$, using the quadratic formula.

Part ii: Subtracting 1 from each term on the left-hand side, and thus subtracting n from the right-hand side, gives

$$\frac{1}{1-x} - 1 + \frac{2}{2-x} - 1 + \cdots + \frac{n}{n-x} - 1 = n - n,$$

which simplifies to

$$\frac{x}{1-x} + \frac{x}{2-x} + \cdots + \frac{x}{n-x} = 0.$$

Factoring out an x reveals that the given equation has the same solutions as equation $(*)$, along with the solution $x = 0$ as well.

Part iii: We first observe that in general the graph of the left-hand side of equation $(*)$ will resemble the one shown above, except that there will be n vertical asymptotes at $x = 1$, $x = 2$, $\ldots$, $x = n$. This explains why there are precisely $n - 1$ solutions, and further shows that $1 < a_1 < 2 < a_2 < 3 < \cdots < n-1 < a_{n-1} < n$. We next show that if a_k is a solution, then $(n+1) - a_k$ is also a solution. Indeed, substituting $x = (n+1) - a_k$ into the left-hand side of equation $(*)$ yields

$$\frac{1}{1-(n+1-a_k)} + \frac{1}{2-(n+1-a_k)} + \cdots + \frac{1}{n-(n+1-a_k)},$$

which may be rewritten as

$$\frac{1}{a_k - n} + \cdots + \frac{1}{a_k - 2} + \frac{1}{a_k - 1}.$$

Since a_k is a solution of $(*)$ this expression equals 0, hence $(n+1) - a_k$ is also a solution. But $k < a_k < k+1$, so $n - k < (n+1) - a_k < n - k + 1$. Hence this solution must be a_{n-k}. In other words, $(n+1) - a_k = a_{n-k}$, or $a_k + a_{n-k} = n + 1$, as claimed. Adding together these equalities for all values of k from 1 to $n - 1$, we discover that

$$2(a_1 + a_2 + \cdots + a_{n-1}) = (n+1)(n-1),$$

meaning that the sum of the solutions is $\frac{1}{2}(n^2 - 1)$.

Part iv: Let a_k be any solution other than the largest. We wish to show that $a_{k+1} - a_k > 1$. Since the graph of the left-hand side of $(*)$ is always increasing, it suffices to show that plugging in $a_k + 1$ gives a negative value, meaning that we have not yet reached the zero in the next interval. We find that

$$\begin{aligned}
&\frac{1}{1-(a_k+1)} + \frac{1}{2-(a_k+1)} + \cdots + \frac{1}{n-(a_k+1)} \\
&\qquad = -\frac{1}{a_k} + \frac{1}{1-a_k} + \cdots + \frac{1}{(n-1)-a_k} \\
&\qquad = -\frac{1}{a_k} - \frac{1}{n-a_k} < 0,
\end{aligned}$$

where we used the fact that a_k is a solution to $(*)$ in the last step.

Part v: For the sake of conserving space we outline the computation. To begin, multiply through by $(1-x)(2-x)\cdots(n-x)$ in $(*)$ and negate if necessary to see that $a_1, a_2, \ldots, a_{n-1}$ are the $n-1$ roots of

$$[(x-2)\cdots(x-n)] + [(x-1)(x-3)\cdots(x-n)] + \cdots + [(x-1)\cdots(x-(n-1))].$$

Let C be the coefficient of the linear term and let D be the constant term. Using Vieta's formulas, one may confirm that the sum of the reciprocals of the roots is equal to $-\frac{C}{D}$. The constant term can be written neatly in the form

$$D = n!(-1)^{n-1}\left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}\right) = n!(-1)^{n-1}A.$$

The linear term is more complicated,

$$C = \frac{n!}{1}(-1)^{n-2}\left(\frac{1}{2}+\cdots+\frac{1}{n}\right)+\cdots+\frac{n!}{n}(-1)^{n-2}\left(\frac{1}{1}+\cdots+\frac{1}{n-1}\right).$$

Every fraction of the form $\frac{1}{jk}$ with $1 \le j < k \le n$ appears twice in the above sum, which means that we can rewrite C as

$$C = n!(-1)^{n-2}\left[\left(\frac{1}{1}+\cdots+\frac{1}{n}\right)^2-\left(\frac{1}{1^2}+\cdots+\frac{1}{n^2}\right)\right],$$

or $C = n!(-1)^{n-2}(A^2-B)$. Finally, simplifying $-\frac{C}{D}$ gives $A-\frac{B}{A}$.

Part vi: We have already seen that $1 < a_1$. To prove that $a_1 < 1+\frac{1}{A}$ it suffices to show that substituting $x = 1+\frac{1}{A}$ into the left-hand side of $(*)$ gives a positive value, since that function is increasing for all x. Since $1+\frac{1}{A} < 2$, this is equivalent to showing that

$$\frac{1}{(1+\frac{1}{A})-1} < \frac{1}{2-(1+\frac{1}{A})}+\frac{1}{3-(1+\frac{1}{A})}+\cdots+\frac{1}{n-(1+\frac{1}{A})}.$$

The left-hand side reduces to just A, which is $1+\frac{1}{2}+\cdots+\frac{1}{n}$. Therefore the above inequality is equivalent to

$$1+\frac{1}{2}+\cdots+\frac{1}{n} < \frac{1}{1-\frac{1}{A}}+\frac{1}{2-\frac{1}{A}}+\cdots+\frac{1}{(n-1)-\frac{1}{A}}.$$

Subtracting every term but $\frac{1}{n}$ over to the right-hand side and combining corresponding pairs of fractions yields

$$\frac{1}{n} < \frac{1}{A-1}+\frac{1}{2(2A-1)}+\cdots+\frac{1}{(n-1)((n-1)A-1)}.$$

However, it is clear that $A-1 < n$, which means that $\frac{1}{n} < \frac{1}{A-1}$. Adding more positive terms to the right-hand side only makes the inequality more pronounced. Hence the latter inequality is true, and you're done.

The interested reader may wish to prove that $a_1^2+\cdots+a_{n-1}^2$ is given by $\frac{1}{12}(n^2-1)(4n+1)$. For a real challenge, try showing that $a_2-a_1 > a_3-a_2$ when $n \ge 5$.

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Round Two Solutions

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