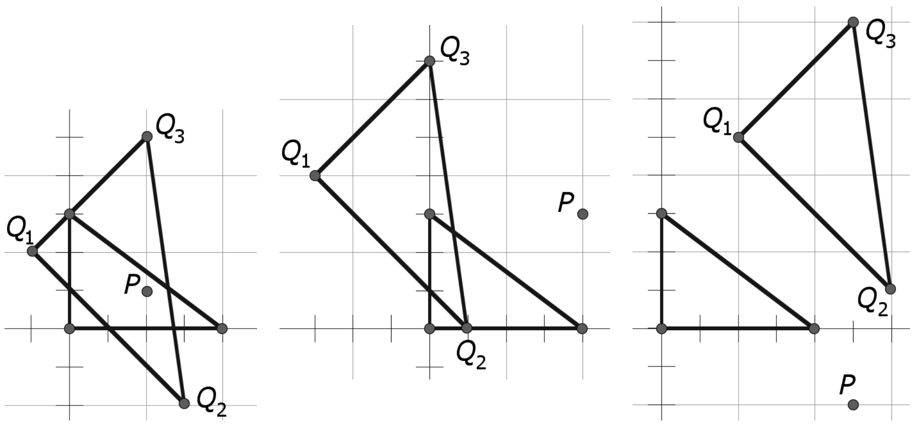
Team Play Solutions

Part i: Using the formula provided we find that the images of $P(2,1)$ have coordinates $Q_1(-1,2)$, $Q_2(3,-2)$ and $Q_3(2,5)$. These coordinates become $Q_1(-3,4)$, $Q_2(1,0)$ and $Q_3(0,7)$ using $P(4,3)$. Meanwhile, the images are $Q_1(2,5)$, $Q_2(6,1)$ and $Q_3(5,8)$ for $P(5,-2)$. These rotations are illustrated in the diagrams below.



All of the image triangles $Q_1Q_2Q_3$ appear to be congruent. More precisely, they are each similar to the original 3–4–5 triangle $A_1A_2A_3$, scaled up by a factor of $\sqrt{2}$ and rotated by $-45°$ from the original.

Part ii: Suppose P is rotated $90°$ about a point A, giving an image Q. (As occurs for P, A_1 and Q_1 in the diagrams above.) We know that $QA = PA$ and that $m\angle PAQ = 90°$. Therefore $\triangle PAQ$ is an isosceles right triangle, with $m\angle QPA = 45°$ and $QP/AP = \sqrt{2}$. We deduce that rotating A by $-45°$ about P and then scaling outwards from P by a factor of $\sqrt{2}$ yields Q. We now realize that in the previous part, instead of rotating P by $90°$ about each of A_1, A_2 and A_3, we can obtain the same resulting images Q_1, Q_2 and Q_3 by rotating $\triangle A_1A_2A_3$ by $-45°$ around P and then scaling outwards from P by $\sqrt{2}$. This explains why all the image triangles were similar to $\triangle A_1A_2A_3$, rotated by $-45°$ and larger by a factor of $\sqrt{2}$.

Part iii: Continuing the reasoning begun in the previous solution, we claim that $area(Q_1Q_2Q_3) = 2area(A_1A_2A_3)$. To see why, let b and h be a base and height of $\triangle A_1A_2A_3$, so that $area(A_1A_2A_3) = \frac{1}{2}bh$. Since $\triangle Q_1Q_2Q_3$ is similar to $\triangle A_1A_2A_3$, just scaled up by a factor of $\sqrt{2}$, we know that the corresponding base and height in $\triangle Q_1Q_2Q_3$ have lengths $b\sqrt{2}$ and $h\sqrt{2}$. Therefore

$$area(Q_1Q_2Q_3) = \frac{1}{2}(b\sqrt{2})(h\sqrt{2}) = bh = 2area(A_1A_2A_3).$$

We know that rotating P by $90°$ about A_1 through A_4 is equivalent to rotating $A_1A_2A_3A_4$ by $-45°$ around P, then scaling up by $\sqrt{2}$. Draw diagonals $\overline{A_1A_3}$ and $\overline{Q_1Q_3}$, splitting each quadrilateral into two triangles. By the above argument, $area(Q_1Q_2Q_3) = 2area(A_1A_2A_3)$ and $area(Q_1Q_3Q_4) = 2area(A_1A_3A_4)$. Since each component of $Q_1Q_2Q_3Q_4$ has twice the area of the corresponding component of $A_1A_2A_3A_4$, the overall ratio of areas will be 2.

Part iv: Let P have coordinates (x,y). Performing the three rotations in the order given yields the images

$$Q_1(3-y, x-3), \qquad Q_2(6-x, 8-y), \qquad Q_3(y-11, 7-x).$$

Since the final image should be $P(x,y)$, we must have $x = y - 11$ and $y = 7 - x$. Solving these equations gives $x = -2$ and $y = 9$, so the unique solution is $P(-2,9)$.

Part v: Suppose the points have coordinates $P_1(x_1,y_1)$ through $P_4(x_4,y_4)$. Rotating P_1 by $90°$ about P_2 yields $(x_2+y_2-y_1, y_2-x_2+x_1)$. Being careful not to drop any negatives during the next two rotations ultimately produces a final image point with coordinates

$$(x_4+y_4+x_3-y_3-x_2-y_2+y_1, -x_4+y_4+x_3+y_3+x_2-y_2-x_1).$$

These coordinates must match $P_1(x_1,y_1)$. Writing the resulting equations in a symmetric form yields

$$\begin{aligned} x_4+y_4+x_3-y_3-x_2-y_2-x_1+y_1 &= 0, \\ -x_4+y_4+x_3+y_3+x_2-y_2-x_1-y_1 &= 0. \end{aligned}$$

We are given that these equations are true, since we are told that these four rotations return P_1 to its original position. On the other hand, in order for P_2 to return to its original position we would need to have

$$\begin{aligned} x_1 + y_1 + x_4 - y_4 - x_3 - y_3 - x_2 + y_2 &= 0, \\ -x_1 + y_1 + x_4 + y_4 + x_3 - y_3 - x_2 - y_2 &= 0. \end{aligned}$$

But these equalities follow from the previous pair; just negate the second equation above and place it on top of the first. This completes the proof.

Part vi: In order to simplify our algebra, we place the origin of our coordinate system at A and orient the x-axis so that it passes through B. Finally, we write the coordinates of B and C as $B(3b, 0)$ and $C(3c, 3d)$. We may now compute the coordinates of the remaining points as

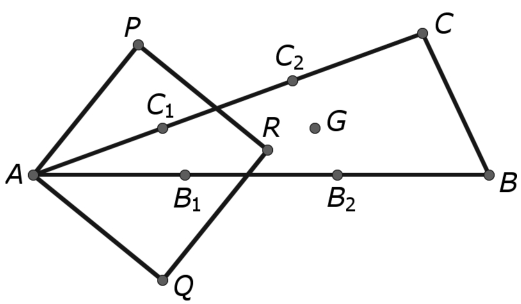
$$B_1(b, 0), \quad B_2(2b, 0), \quad C_1(c, d), \quad C_2(2c, 2d), \quad G(b + c, d).$$

The latter expression comes from the fact that the centroid G of $\triangle ABC$ is found by averaging the coordinates of A, B and C.

Using our formula for 90° rotations, we also find

$$A(0, 0), \quad P(b - d, c), \quad Q(c, -b + d), \quad R(b + c - d, -b + c + d).$$

To show that $APRQ$ is a square it suffices to demonstrate that rotating Q by 90° about A gives P and also that rotating A by 90° about P gives R. (The reader should confirm this fact.) But it is straight-forward to find that these images have coordinates $(b-d, c)$ and $(b+c-d, -b+c+d)$, which match the coordinates of P and R, as desired.



The Mandelbrot Team Play

Round 2010 Solutions

Greater Testing Concepts
PO Box 760
Potsdam, NY 13676

The Mandelbrot Team Play
www.mandelbrot.org
info@mandelbrot.org