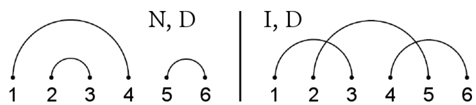
Team Play Solutions

Part i: Given a row of six dots, the leftmost dot may be connected to any of the other five. Then the leftmost of the remaining dots may be paired up in three ways, leaving two dots which must be paired with one another, for a total of $5 \cdot 3 \cdot 1 = 15$ possible pairings. To conserve space we only draw two of these pairings; they are starred in the list below.



Pairing	Label
(1,2) (3,4) (5,6)	D
(1,2) (3,5) (4,6)	I, D
(1,2) (3,6) (4,5)	N, D
(1,3) (2,4) (5,6)	I, D
(1,3) (2,5) (4,6)	I, D*
(1,3) (2,6) (4,5)	I, N, D
(1,4) (2,3) (5,6)	N, D*
(1,4) (2,5) (3,6)	I

Pairing	Label
(1,4) (2,6) (3,5)	I, N
(1,5) (2,3) (4,6)	I, N, D
(1,5) (2,4) (3,6)	I, N
(1,5) (2,6) (3,4)	I, N
(1,6) (2,3) (4,5)	N, D
(1,6) (2,4) (3,5)	I, N
(1,6) (2,5) (3,4)	N

Part ii: By examining the list in the previous solution, we find exactly 6 pairings which are not labeled with a 'D.' In order to obtain such a pairing when $n = 4$, split the eight dots down the middle to obtain four left-hand dots and four right-hand dots. Then pair the left-hand dots with the right-hand dots in any manner. No two semicircles drawn in this way can be disjoint, since the right endpoint of the first semicircle must be located past (to the right of) the left endpoint of the other semicircle, causing them to either intersect or be nested. Since the right endpoint of every semicircle must be located past every left endpoint, all pairings not involving disjoint semicircles arise in this manner.

This construction works in general, so such a pairing of $2n$ dots can be done in $n!$ ways, since the first left-hand dot can be paired with any of the n right-hand dots, then the second left-hand dot can be paired with any of the remaining $n-1$ right-hand dots, and so on.

Part iii: It is possible to create an exhaustive list of all such pairings; however, we will describe a more efficient approach here. We choose to organize our count by considering each dot labeled R, moving from left to right. The first such R can be connected to any of the three L's that precede it. Regardless of how we draw this semicircle, the next R may be joined to any of the remaining two L's, then there are once again two L's which could pair with the third R (since we pass over an L on the way), and so on. The grand total is $(3)(2)(2)(2)(1) = 24$ ways.

Observe that it is not helpful to consider L's rather than R's since the number of options for drawing semicircles starting at the third L will vary depending upon how the first two semicircles are drawn. In particular, we are forced to connect that L to the R at its right if that R has not already been taken.

Part iv: We claim that it is possible to create a pairing that follows the labeling if and only if the number of R's never exceeds the number of L's as we tally the letters from left to right. Clearly this condition is necessary, for if the number of R's ever did exceed the number of L's within some initial portion of dots, then there would not be enough left endpoints of semicircles to pair with those right endpoints. On the other hand, it is always possible (usually in many ways) to create such a pairing if the R's never excced the L's as we move from left to right. Just move from one R to the next, drawing a semicircle back to some L that came before it. There will always be at least one L available by our condition, which guarantees that we can finish the pairing.

Part v: As in the previous solution, let us consider our options as we move from one dot labeled R to the next, moving from left to right. We claim that at each step we are forced to join that R dot to the *nearest* of all the available L dots to its left. For if draw this semicircle over an available L dot, then that dot will have to be joined to an R dot further on, causing an intersection. According to the previous solution there will always be at least one L dot available, so we can complete this construction to obtain a pairing with no intersections. (The reader should perform this process for the labeling given in Part iii.)

Part vi: The final part brings us to the real goal of these questions; namely, to show that there are as many pairings without intersections as there are without nesting. Why should one expect *a priori* for these pairings to be equinumerous, as opposed to the pairings with no disjoint semicircles? We are not entirely sure.

So consider again a labeling of $2n$ points for which there are pairings that follow the labeling. We claim that for any such labeling there is also exactly one pairing that does not involve any nested semicircles. As before, step from one dot labeled R to the next, moving from left to right. We claim that at each step we are now forced to join that R dot to the *furthest* of all the available L dots to its left. For if this semicircle stops short of an available L dot, then that dot must be joined to an R dot further on, creating nested semicircles. As usual, it is always possible to complete this construction, yielding exactly one pairing without nesting.

In summary, for each labeling of the $2n$ points for which the number of R's never exceeds the number of L's moving from left to right, there is precisely one pairing with no intersections and one pairing with no nesting. Hence there are the same number of each type, and we're done.

We are grateful to Gabriel Carroll for bringing to our attention the proof of this result on which these questions are based.

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Round Two Solutions

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