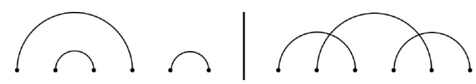
Team Play Topics

Round Two

The first section of the Round Two Mandelbrot Team Play is reproduced below. A list of topics and practice problems are also provided to aid in preparation. Note that these problems are not meant to serve as a precise indicator of the problems that will appear on the contest. However, students who understand how to solve them should be able to make significantly more progress than they might have otherwise. So work hard on the problems, and good luck on the Team Play!

Definitions: Let n be a positive integer. Given $2n$ equally spaced dots in a horizontal row, a *pairing* of the points consists of splitting the points into n pairs, then drawing n semicircles above the row joining each pair of dots. Two different pairings for $n = 3$ are pictured below. Semicircles are situated relative to one another in three different ways: they could *intersect* (if they cross over one another), they could be *nested* (if one is contained completely within the other), or else they could be *disjoint* (if their interiors do not overlap). Thus the pairing on the left involves nested and disjoint semicircles, but no intersection. The pairing on the right includes intersecting and disjoint semicircles, but no nesting.



Topics: Pairings of $2n$ points, elementary counting techniques, bijections

Practice Problems

1. Find a pairing for $n = 3$ that involves two intersections, one nesting, and no disjoint semicircles.

2. Find a pairing in the case $n = 4$ that involves three sets of disjoint semicircles, two nestings, and one intersection.

3. Given a value of n, how many pairings involve *only* disjoint semicircles? Draw each such pairing.

4. Continuing the previous problem, now do the same for pairings that involve only nested semicircles, and finally for pairings that involve only intersecting semicircles.

5. Count the number of pairings that do not involve intersections in the cases $n = 1$, 2, 3, and 4. Draw the pairings in each case.

6. How many pairings are there when $n = 5$ that do not involve intersections? (Tip: split the count into cases according to which point the leftmost dot is paired with, then use your answers from the previous problem.)

Hints and answers on the next page. $\Longrightarrow$

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Hints and Answers

1. Number the dots 1 to 6 from left to right, then use semicircles to connect the pairs $(1, 4)$, $(2, 6)$, and $(3, 5)$. Another possible answer is the mirror reflection of this pairing.

2. Number the dots 1 to 8 from left to right, then use semicircles to connect the pairs $(1, 5)$, $(2, 6)$, $(3, 4)$ and $(7, 8)$. Another possible answer is the mirror reflection of this pairing.

3. There is but a single way to obtain only disjoint semicircles. This may be accomplished by creating the pairing $(1, 2)$, $(3, 4)$, $\ldots$, $(2n - 1, 2n)$.

4. Just as before the pairings are unique. To obtain only nested semicircles join the dots numbered $(1, 2n)$, $(2, 2n-1)$, $\ldots$, $(n, n+1)$. To obtain only intersecting semicircles use the pairing $(1, n+1)$, $(2, n + 2)$, $\ldots$, $(n, 2n)$.

5. There are exactly 1, 2, 5 and 14 pairings which do not involve intersecting semicircles in the cases $n = 1$, 2, 3 and 4. We omit the drawings of these pairings.

6. If we use the pairing $(1, 2)$, then the remaining eight dots must be connected without intersections, which can be done in fourteen ways according to the previous answer. We cannot use the pairing $(1, 3)$, since then dot 2 would create an intersection. (In general, we cannot leave an odd number of dots underneath the first semicircle.) If we use the pair $(1, 4)$ then there is one way to pair up dots 2 and 3, and five ways to pair up dots 5 through 10. Continuing in this manner, we eventually pair up $(1, 10)$, leaving fourteen ways to pair up dots 2 through 9. Our overall count comes to

$$14 + (1)(5) + (2)(2) + (5)(1) + 14 = 42.$$

The reader may recognize the sequence 1, 2, 5, 14, 42, $\ldots$ as the sequence of Catalan numbers. While this topic is lovely and worth delving into further, the upcoming Team Play questions do *not* directly pertain to Catalan numbers.