Answer Key

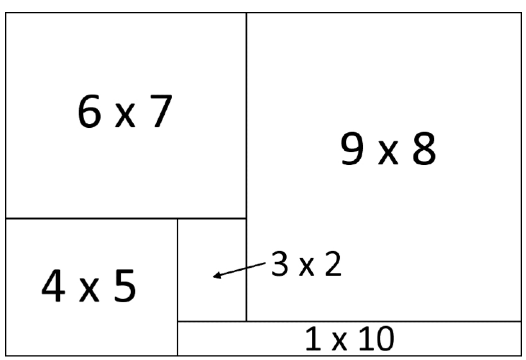
1.	42	4.	9
2.	10×15	5.	1707
3.	$x^2 + 5x + 5$	6.	bb
		7.	10

1. Working backwards, we figure that there must have been exactly three cherries left after Yanson ate half of them, meaning there were six cherries in the bowl when he found the bowl. In the same fashion there must have been 18 cherries in the bowl when Meena began, and finally **42** cherries in the bowl to start with.

2. The total area of the large rectangle is

$$1 \cdot 10 + 2 \cdot 3 + 4 \cdot 5 + 6 \cdot 7 + 8 \cdot 9 = 150.$$

Therefore the length and width of the large rectangle are positive integers that multiply to 150. Furthermore, both the length and width must be at least 8, in order to accommodate the 8×9 rectangle. The only rectangle satisfying these conditions has dimensions $\mathbf{10 \times 15}$. One such arrangement is shown below.



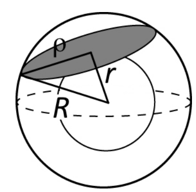
3. Likely the fastest way to obtain the answer is by trial and error. Before long one discovers that $\boldsymbol{x^2 + 5x + 5}$ does the job, since

$$x^2 + 5x + 6 = (x+2)(x+3), \qquad x^2 + 6x + 5 = (x+1)(x+5).$$

The interested reader may wish to determine other values of b and c with the given property. One would need both $(b+1)^2 - 4c$ and $b^2 - 4(c+1)$ to be perfect squares. Observe that the former is $2b+5$ greater than the latter. Two squares differing by $2b+5$ could be $(b+2)^2$ and $(b+3)^2$, leading to $b+c = -2$. Or if b is of the form $b = 3k+2$ then the squares k^2 and $(k+3)^2$ differ by $6k+9 = 2b+5$ again, leading to $c = 2k^2+3k$. (The above solution is of this type, with $k = 1$.) Can you find all solutions?

4. The third statement allows us to deduce that s must be odd. For if s were an even digit (i.e. either 2, 4, 6, or 8) then s^2 would end with a 4 or 6, implying that t^2 would end with a 3 or 7. But this is impossible; no square ends with a 3 or 7. Now that we know s is odd the first statement implies that t is spelled with four letters; hence t equals four, five or nine. But t cannot be spelled with an 'f' since otherwise the second statement would imply that s is even, which can't happen. This rules out four and five, leaving $t = \mathbf{9}$.

5. In order to make any progress it is necessary to recall that the surface area of a sphere with radius R is given by $4\pi R^2$. So let R be the radius of the large sphere, let r be the radius of the small sphere, and let ρ the radius of the shaded circle. Drawing in all these radii as shown creates a right triangle (because the shaded circle is tangent to the small sphere), thus $r^2 + \rho^2 = R^2$ according to the Pythagorean Theorem. Multiplying through by 4π will be convenient, yielding



$$4\pi r^2 + 4\pi\rho^2 = 4\pi R^2 \qquad \Longrightarrow \qquad 4\pi r^2 = 2011 - 4(76) = 1707.$$

We conclude that the surface area of the large sphere is **1707**. Give yourself a pat on the back if you also realized that this is the year in which Euler was born!

6. The number of b's either stays constant or decreases by five for any given move. There are seven b's initially, so the shortest string of letters we could obtain, in theory, is bb. We now show that this is indeed possible. (We abbreviate aaa as a^3 and similarly.)

$$
\begin{aligned}
abababababababab \;\rightarrow\; ba^3babababababab \;\rightarrow\; ba^3b^3a^7babab \\
\rightarrow\; ba^3b^2a^3bababab \;\rightarrow\; \cdots \;\rightarrow\; ba^3b^5a^{31}b
\end{aligned}
$$

Erasing the b^5 now leaves us eventually with $baab$, but we can do better. The reader should confirm that moving an a "past the b^5" yields b^5a^{32}. Doing this twice converts the above string to $bab^5a^{95}b$. We now erase the b^5 to get $ba^{96}b$, which reduces to just $\boldsymbol{bb}$, as desired.

7. Let E, F and G represent the expected number of moves to reach the starred vertex from the circled vertex, from a vertex adjacent to the circle, and from a vertex adjacent to the star, respectively. Then the following relationships exist among E, F and G.

$$
\begin{aligned}
E &= F + 1 \\
F &= \tfrac{2}{3}G + \tfrac{1}{3}E + 1 \\
G &= \tfrac{2}{3}F + 1
\end{aligned}
$$

The middle equation arises from the fact that if we are currently at a vertex next to the circle, then $\frac{2}{3}$ of the time we will move to a vertex next to the star, while $\frac{1}{3}$ of the time we will land back on the circle. Hence the expected number of moves from there is $\frac{2}{3}G$ added to $\frac{1}{3}E$, plus the 1 move it takes to get there. We arrive at the other equations in the same manner. Solving is routine, yielding $E = \mathbf{10}$.

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