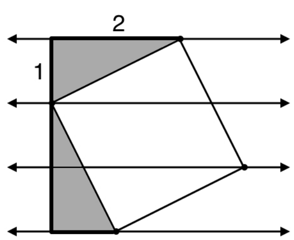
Answer Key			
1.	\$4	4.	7
2.	39	5.	10
3.	5	6.	1.5
		7.	39

1. If we suppose the \$20 bill still belongs to Vincent when it is dropped, then Melissa owes him \$10 in order to share the loss. Combining this with Vincent's debt of \$14, we conclude that he has to give Melissa **\$4**. On the other hand, if we suppose that the \$20 bill belongs to Melissa when it is dropped then Vincent has to give her another \$10 to share the loss, but is owed \$6 in change for his original \$14 debt, so regardless he has to give Melissa **\$4**.

2. We claim that the largest possible value for Drake's number is 39. This value is certainly possible, for the three numbers could be 39, 40 and 50, each with a different tens digit and having a product of 78,000. But if Drake's number is 40 or more then the next highest number must be at least 50, and the largest number must be at least 60, for a product of at least 120,000, which is too large. Hence **39** is the maximal value.

3. The four equally spaced horizontal lines with the inscribed square are shown here. Observe that rotating the square 90° clockwise about its center will return the square to the same place, but move the lower left shaded triangle to the position of the upper left shaded triangle. Hence the horizontal leg of the upper triangle is the same as the vertical leg of the lower triangle, which is 2. Therefore the hypotenuse is $\sqrt{5}$ by the Pythagorean Theorem, so the area of the square is $(\sqrt{5})^2 = \mathbf{5}$.



4. Suppose we were to select the number 3 in advance. Then the possible products we could obtain would be 3, 6, 9, 12, 15, 18, 21 or 24. Each units digit is equally likely, so on average we will win the amount

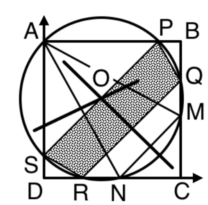
$$\frac{1}{8}(3+6+9+2+5+8+1+4) = 4.75.$$

Performing a similar computation for all the numbers from 1 to 8 yields expected winnings of 4.5, 4, 4.75, 4.25, 2.5, 4.5, 5.25 and 4.75 with the highest average when we choose **7**.

5. For convenience, let's call call the seven people at the party who each know seven others the Pontipee brothers. Since each Pontipee only has six brothers, each of them must know at least one person other than another Pontipee. This fact forces an eighth person to be at the party. This person knows at most five of the Pontipees, but all seven need to know at least one other party-goer, so we must have a ninth person as well. And we can't stop here, since this would entail $7(7) + 2(5) = 59$ instances where one person knows another, when the total must be even since knowing is mutual, bringing the minimum up to ten.

In fact it is possible to arrange such a party with ten people: call the Pontipees P_1 through P_7, and let each of them know all the others except for P_6 and P_7, who know the first five but don't know each other. Call the remaining three people at the party the Querklin sisters Q_1, Q_2 and Q_3, who all know each other. Finally, let Q_1 know P_1, P_2 and P_3; let Q_2 know P_4, P_6 and P_7; and let Q_3 know P_5, P_6 and P_7. This satisfies all the conditions, so **10** is the answer.

6. We reproduce the diagram at right for reference. Set up a coordinate system whose origin is at D. The midpoint of $\overline{AN}$ is $(.5, 1)$, hence the perpendicular bisector of $\overline{AN}$ has equation $y = \frac{1}{2}x + \frac{3}{4}$. Meanwhile, the perpendicular bisector of $\overline{MN}$ has equation $y = 2 - x$. The center of the circle lies at the intersection of these two lines, namely $(\frac{5}{6}, \frac{7}{6})$. Since M and Q are situated symmetrically above and below the center, we deduce that Q has coordinates $(2, \frac{4}{3})$. Similarly, A and P are the same distance to the left and right



of the center, so P has coordinates $(\frac{5}{3}, 2)$. In the same manner we find $R(\frac{2}{3}, 0)$ and $S(0, \frac{1}{3})$. It is now straight-forward to subtract the areas of the four triangular regions surrounding $PQRS$ from the area of the square to find that

$$\text{area}(PQRS) = 4 - \tfrac{1}{2}(\tfrac{5}{3})(\tfrac{5}{3}) - \tfrac{1}{2}(\tfrac{4}{3})(\tfrac{4}{3}) - \tfrac{1}{2}(\tfrac{1}{3})(\tfrac{2}{3}) - \tfrac{1}{2}(\tfrac{2}{3})(\tfrac{1}{3}).$$

This reduces to $4 - \frac{45}{18} = \frac{3}{2}$, so area$(PQRS) = \mathbf{1.5}$.

7. We observe that $g(n) = 2$ if and only n is a prime, since composite numbers all have at least one proper divisor between 1 and n, making $g(n)$ larger than 2. Therefore $g(g(n)) = 2$ exactly when $g(n)$ is a prime. We now count how many times this occurs for $2 \le n \le 100$. Evidently we could have $g(n) = 2$ or $g(n) = p$ for an odd prime $p > 2$. In the first case n itself must be prime (as we have just seen), and there are 25 primes from 2 to 100. But there are also cases such as $n = 44$ whose largest proper divisor 22 is one less than a prime, making $g(44) = 23$. These cases occur whenever n has the form $n = 2(p-1)$ for p an odd prime. There are 14 such values of n, from $2(3-1) = 4$ through $2(47-1) = 92$, for a grand total of $25 + 14 = \mathbf{39}$ values of n.

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