January 2013

The Mandelbrot Team Play

Round One Test

Time Limit: 60 minutes

Facts: given a sequence $a_0, a_1, a_2, a_3, a_4, \ldots$, of numbers one can use *finite differences* to find a polynomial formula for the sequence. Compute the differences $b_0 = a_1 - a_0$, $b_1 = a_2 - a_1$, $b_2 = a_3 - a_2$, and so forth, writing these values in a row beneath the first. Next compute a second row of differences $c_0 = b_1 - b_0$, $c_1 = b_2 - b_1$, $c_2 = b_3 - b_2, \ldots$, and continue this process until some row contains all 0's. Then the desired formula is

$$a_n = a_0 + b_0 n + c_0 \binom{n}{2} + d_0 \binom{n}{3} + \cdots = a_0 + b_0 n + c_0 \frac{n(n-1)}{2 \cdot 1} + d_0 \frac{n(n-1)(n-2)}{3 \cdot 2 \cdot 1} + \cdots.$$

Note that if the sequence begins at a_1 or later then one should extend the terms of the rows backwards, working from bottom to top, to obtain the values of a_0, b_0, c_0, d_0, etc.

Setup: Suppose that a, b and c are positive integers for which $a^2 + b^2 + c^2$ is divisible by $abc + 1$. In this case we call the trio of numbers (a, b, c) an *excellent* triple and say that its q-value is the value of the quotient $q = (a^2 + b^2 + c^2)/(abc + 1)$. In the questions below you will investigate the remarkable claim that the set of q-values arising from excellent triples are precisely those numbers that can be written as the sum of two positive integer squares.

Problems

Part i: (4 points) Demonstrate that the triples $(1, 1, 2)$, $(1, 2, 3)$, $(1, 3, 4)$ and $(1, 4, 5)$ are all excellent. Compute the q-value for each trio and show that it is the sum of two squares.

Part ii: (4 points) Based on the pattern begun in the previous part, show that there are infinitely many excellent triples whose smallest element is 1.

Part iii: (5 points) It is a fact that $(2, 1, 10)$, $(2, 2, 32)$, $(2, 3, 78)$, $(2, 4, 160)$ and $(2, 5, 290)$ are all excellent triples. Use this information to find a formula that generates infinitely many triples beginning with 2, confirm that these triples are excellent, and find their q-values.

Part iv: (5 points) Suppose that (a, b, c) is an excellent triple having quotient q. Prove that $a^2 + b^2 + (qab - c)^2$ is divisible by $ab(qab - c) + 1$.

Part v: (5 points) Let (a, b, c) be an excellent triple with $a \le b \le c$, having quotient q. Prove that $qab - c \ge 0$. If in addition we have $q \ge 2$, then prove that $qab - c < b$ as well.

Part vi: (5 points) Explain why the q-value associated with any excellent triple (a, b, c) must be equal to the sum of two positive integer squares, as long as $q \ge 2$.