ANSWER KEY	
1. 47	4. 99
2. 3891676	5. 25/4
3. 3/2	6. 291
	7. 254

1. It may be tempting to solve the equation; however, its solutions contain radicals that are hard to work with. Here is a cleaner way. First expand $(x+3)(x+4)$ to get $x^2+7x+12$. We are given $x^2+7x=35$, so adding 12 to both sides gives $x^2+7x+12=\mathbf{47}$, which is our answer.

2. Note that the numbers 3, 6, 6, 9 are all divisible by 3, so they must occupy the first, third, fifth and seventh squares. Also, 8 must not be adjacent to either of the 6s, which restricts placement of the 6s to one side or the other. To minimize our number, we try placing the smallest possible numbers from left to right. The smallest number that can be placed in the first square is 3. This rules out the 6s being on the left side, so they must be on the right, and thus 8 is the second digit. The rest of the squares can easily be deduced, with the 1 preceding the 7, and the resulting number is **3891676**.

3. Using addition of logarithms, we obtain

$$(\log_{28} 216)(\log_6 2 + \log_6 \sqrt{7}) = (\log_{28} 216)(\log_6 2\sqrt{7})$$

Note that $2\sqrt{7} = \sqrt{28} = 28^{1/2}$, so $\log_6 2\sqrt{7} = \log_6 28^{1/2} = \frac{1}{2}\log_6 28$. Finally, using the logarithm change-of-base formula, we get

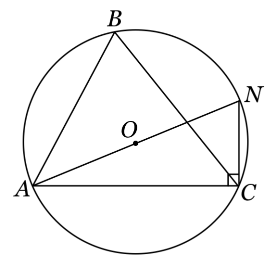
$$(\log_{28} 216)(\tfrac{1}{2}\log_6 28) = \tfrac{1}{2}(\log_6 28)(\log_{28} 216) = \tfrac{1}{2}\log_6 216 = \mathbf{\frac{3}{2}}.$$

4. Let the number of books and bookmarks Lillie receives be a and b, respectively; thus the number of ways to choose a book and bookmark is ab. The number of books and bookmarks Selene has left is $25-a$ and $16-b$, giving $(25-a)(16-b)$ ways for her to choose a book and bookmark. The problem can thus be written algebraically as

$$ab = (25-a)(16-b)+1.$$

Expanding and simplifying gives $16a+25b=401$, at which point guess-and-check by varying b is an efficient way to finish the problem. Trying out values of b up to 16 yields the only positive integer set of solutions $a=11$, $b=9$. Therefore the number of ways that Lillie can choose a book and bookmark is $11\cdot 9 = \mathbf{99}$.

5. In the diagram below we have drawn diameter $\overline{AN}$ and segment $\overline{CN}$. Since an angle inscribed in a semicircle always measures $90°$, we also label $\angle ACN$ as a right angle. This immediately suggests a solution strategy; if we could determine length AN, then we could find CN via the Pythagorean Theorem.



A standard method to find the diameter of a circumcircle is to combine Hero's area formula with the circumradius area formula, as follows. The semiperimeter is $s=\frac{1}{2}(13+14+15)=21$, hence

$$K = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{(21)(8)(7)(6)} = 84,$$

where K represents *area*(ABC). But we also know that

$$K = \frac{abc}{4R} \qquad \Longleftrightarrow \qquad R = \frac{13\cdot 14\cdot 15}{4\cdot 84} = \frac{65}{8}.$$

We deduce that the diameter length $AN = 2R = \frac{65}{4}$. We could now finish via Pythagorean, but it's quicker to observe that $\triangle ACN$ is simply a 5–12–13 triangle scaled by a factor of $\frac{5}{4}$ up to a $\frac{25}{4}$–15–$\frac{65}{4}$ right triangle. Hence the answer is $CN = \mathbf{\frac{25}{4}}$.

6. An equation with two remainder functions might seem complicated. However, we can exploit the fact that if $a<b$ then $r(a,b)=a$, while if $a>b$ then $r(b,a)=b$; either way we can eliminate one of the remainder functions. So we split this problem into cases.

Case 1: $a < b$. In this case we have $r(a, b) = a$, so the equation becomes $a + r(b, a) = a$, or $r(b, a) = 0$. Thus b must be a multiple of a.
Case 2: $a = b$. Here both $r(a, b)$ and $r(b, a)$ are 0, which cannot add up to a, which is positive. Hence there are no solutions in this case.
Case 3: $a > b$. Then $r(b, a) = b$, so the equation becomes $r(a, b) + b = a$, or $b + r(a, b) = a$. This means that b goes into a once, and what is left over is the remainder. In other words, we must have $a < 2b$.

One can now find the number of pairs (a, b) such that $1 \leq a, b \leq 30$. It is helpful to note that there are no solutions with $15 < a < b$, and the number of solutions in the case $a > b$ follows a linear pattern as b increases. All together there are **291** solutions.

7. Let E_n be the expected number of steps for Humpty Dumpty to get to the top from rung n, for $0 \leq n \leq 7$, where the ground is rung 0, and the top is rung 7. Then for rungs 0 to 6, when he takes a step, he is equally likely to go up one rung as to fall to the ground (rung 0). Thus

$$E_k = \frac{1}{2}E_0 + \frac{1}{2}E_{k+1} + 1.$$

When he reaches rung 7 he is done, so $E_7 = 0$. We now have a system of equations, which we are solving for E_0. Note that the above equation gives E_k in terms of E_0 and E_{k+1}. By starting with $E_0 = \frac{1}{2}E_0 + \frac{1}{2}E_1 + 1$ and substituting all E_k (for $k > 0$) with $\frac{1}{2}E_0 + \frac{1}{2}E_{k+1} + 1$ up to E_7, we ultimately obtain a single equation with just E_0. This process yields

$$E_0 = \frac{127}{128}E_0 + \frac{1}{128}E_7 + \frac{127}{64} = \frac{127}{128}E_0 + \frac{127}{64},$$

which we solve to get $E_0 = \mathbf{254}$.

Problem Credits

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⋆ National Level ⋆

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