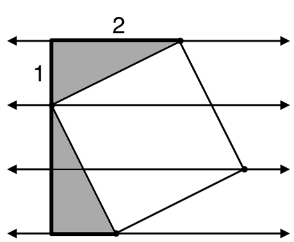
Answer Key	
1. \$4	4. 10
2. 39	5. 39
3. 5	6. 7
	7. 14/125

1. If we suppose the \$20 bill still belongs to Vincent when it is dropped, then Melissa owes him \$10 in order to share the loss. Combining this with Vincent's debt of \$14, we conclude that he has to give Melissa **\$4**. On the other hand, if we suppose that the \$20 bill belongs to Melissa when it is dropped then Vincent has to give her another \$10 to share the loss, but is owed \$6 in change for his original \$14 debt, so regardless he has to give Melissa **\$4**.

2. We claim that the largest possible value for Drake's number is 39. This value is certainly possible, for the three numbers could be 39, 40 and 50, each with a different tens digit and having a product of 78,000. But if Drake's number is 40 or more then the next highest number must be at least 50, and the largest number must be at least 60, for a product of at least 120,000, which is too large. Hence **39** is the maximal value.

3. The four equally spaced horizontal lines with the inscribed square are shown here. Observe that rotating the square 90° clockwise about its center will return the square to the same place, but move the lower left shaded triangle to the position of the upper left shaded triangle. Hence the horizontal leg of the upper triangle is the same as the vertical leg of the lower triangle, which is 2. Therefore the hypotenuse is $\sqrt{5}$ by the Pythagorean Theorem, so the area of the square is $(\sqrt{5})^2 = \mathbf{5}$.



4. For convenience, let's call call the seven people at the party who each know seven others the Pontipee brothers. Since each Pontipee only has six brothers, each of them must know at least one person other than another Pontipee. This fact forces an eighth person to be at the party. This person knows at most five of the Pontipees, but all seven need to know at least one other party-goer, so we must have a ninth person as well. And we can't stop here, since this would entail $7(7) + 2(5) = 59$ instances where one person knows another, when the total must be even since knowing is mutual, bringing the minimum up to **10**.

In fact it is possible to arrange such a party with ten people: call the Pontipees P_1 through P_7, and let each of them know all the others except for P_6 and P_7, who know the first five but don't know each other. Call the remaining three people at the party the Querklin sisters Q_1, Q_2 and Q_3, who all know each other. Finally, let Q_1 know P_1, P_2 and P_3; let Q_2 know P_4, P_6 and P_7; and let Q_3 know P_5, P_6 and P_7.

5. We observe that $g(n) = 2$ if and only n is a prime, since composite numbers all have at least one proper divisor between 1 and n, making $g(n)$ larger than 2. Therefore $g(g(n)) = 2$ exactly when $g(n)$ is a prime. We now count how many times this occurs for $2 \leq n \leq 100$. Evidently we could have $g(n) = 2$ or $g(n) = p$ for an odd prime $p > 2$. In the first case n itself must be prime (as we have just seen), and there are 25 primes from 2 to 100. But there are also cases such as $n = 44$ whose largest proper divisor 22 is one less than a prime, making $g(44) = 23$. These cases occur whenever n has the form $n = 2(p - 1)$ for p an odd prime. There are 14 such values of n, from $2(3 - 1) = 4$ through $2(47 - 1) = 92$, for a grand total of $25 + 14 = \mathbf{39}$ values of n.

6. There is a delightfully quick way to find the correct answer. Observe that it is not possible to reach the star on the first roll, but on every move after that there is precisely one outcome from the die roll that results in landing on the star. So after the first roll there is a $\frac{1}{6}$ probability of success on each turn. A standard expected value fact states that in this situation it will take six rolls on average to achieve the desired outcome. Adding the initial roll brings us to a total of **7** rolls on average. We can

confirm this fact by letting E be the expected number of rolls (after the initial roll) to reach the star. Then $E = \frac{1}{6}(1) + \frac{5}{6}(E+1)$, since $\frac{1}{6}$ of the time we land on the star on our first try, while $\frac{5}{6}$ of the time we're back where we started, with one roll already made. Solving gives $E = 6$.

7. Since $\theta_1 + \theta_2 + \theta_3 = 360°$, by taking sine of both sides it follows that $\sin\theta_1 \sin\theta_2 \sin\theta_3$ is equal to

$$\sin\theta_1 \cos\theta_2 \cos\theta_3 + \cos\theta_1 \sin\theta_2 \cos\theta_3 + \cos\theta_1 \cos\theta_2 \sin\theta_3.$$

Similarly, taking cosine of both sides shows $\cos\theta_1 \cos\theta_2 \cos\theta_3 - 1$ equals

$$\cos\theta_1 \sin\theta_2 \sin\theta_3 + \sin\theta_1 \cos\theta_2 \sin\theta_3 + \sin\theta_1 \sin\theta_2 \cos\theta_3.$$

We are given that $\cos\varphi = \frac{4}{5}$, so $\sin\varphi = \frac{3}{5}$. We now use the fact that $\cos(\varphi + \theta_j) = \frac{4}{5}\cos\theta_j - \frac{3}{5}\sin\theta_j$ to expand the desired product into a sum of eight terms. Using the two identities just derived, the entire sum boils down to just

$$\begin{aligned}\cos(\varphi+\theta_1)\cos(\varphi+\theta_2)\cos(\varphi+\theta_3) \quad &= \quad \cos\varphi\cos\theta_1\cos\theta_2\cos\theta_3 - \\ &\qquad \sin\varphi\sin\theta_1\sin\theta_2\sin\theta_3 - \frac{36}{125}\end{aligned}$$

We leave the reader the pleasurable task of supplying the details. Using the information given in the problem, we can now finish the computation to see that the value is $\frac{2}{5} - \frac{36}{125} = \mathbf{\frac{14}{125}}$.

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PO Box 760
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The Mandelbrot Competition
www.mandelbrot.org
info@mandelbrot.org