Team Play Solutions

Part i: We find that there are a total of eight ordered partitions of 4:

$$4, \quad 3+1, \quad 1+3, \quad 2+2, \quad 2+1+1$$
$$1+2+1, \quad 1+1+2, \quad \text{and} \quad 1+1+1+1.$$

Turning each sum into a product and adding up the results gives

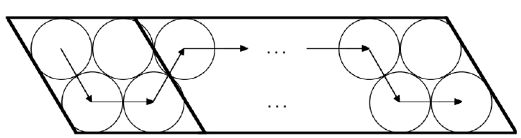
$$4+3+3+4+2+2+2+1=21.$$

As anticipated, 21 is a Fibonacci number, since the Fibonacci sequence begins 1, 1, 2, 3, 5, 8, 13, 21, etc. (By the way, it is merely a coincidence that there were eight ordered partitions and 8 is a Fibonacci number. There are sixteen ordered partitions of 5, for example.)

Part ii: Since we are not permitted to move up/right, we are forced to make a single down/right step, with the remaining moves being purely to the right. In two rows of n circles there are exactly n locations where the down/right move could occur, leading to a total of n ways to create a path with the given restriction.

The up/right move at the fourth step splits the path into two shorter paths of the type just discussed. We string together a path through two rows of 3 circles with only right and down/right moves, followed by the up/right step, followed by the remainder of the path through two rows of $n-3$ circles with only right and down/right moves. Hence there are a total of $3(n-3)$ such paths.

Part iii: As just explained, each up/right step splits the rows of n circles into two shorter pairs of rows (whose lengths sum to n) which we may traverse sequentially. For instance, the up/right step below divides the rows into two shorter pairs of rows of lengths 2 and $n-2$.



This subdivision may be repeated several times if there is more than one up/right step. Therefore each group of paths, sorted according to the location of the up/right steps, corresponds to exactly one ordered partition of n; namely, the partition of n into the blocks created by the up/right steps. Thus the diagram above corresponds to the ordered partition $2+(n-2)$. In particular, the number of such groups will be equal to the number of ordered partitions of n.

Part iv: We now consider how many paths have up/right steps at specified locations. As we have seen, these up/right steps split the rows of n circles into several smaller blocks, each of which is traversed from the upper left to the lower right circle using only right and down/right moves. And in a previous part we also demonstrated that the number of ways to traverse a block in this manner is equal to the length of the block. Hence the total number of such paths is equal to the product of the lengths of the blocks.

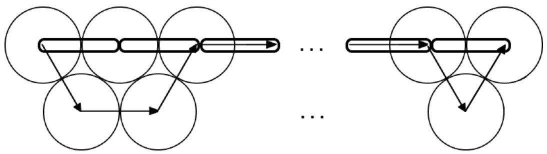
In other words, if we sort all paths into groups according to the location of the up/right steps then we obtain one group for each ordered partition of n, and the number of paths within this group is given by the product of the terms in that particular ordered partition. Adding up all these products gives the total number of paths, which is known to be F_{2n} by the Facts section. This completes the explanation.

Part v: Although a proof involving path counting is possible, we opt for a more algebraic explanation. Consider the set of all ordered partitions of n that have a particular number b as the last term. Erasing the final b gives all ordered partitions of $n-b$, so adding these ordered partition products yields $F_{2(n-b)}$. Since changing the b to a 1 gives the same products, we still obtain $F_{2(n-b)}$. Therefore the sum over all ordered partitions of n will be

$$F_{2n-2}+F_{2n-4}+\cdots+F_4+F_2+1.$$

The final one comes from the ordered partition consisting of just n by itself. But $1+F_2=F_3$, then $F_3+F_4=F_5$, and so on. Hence the entire sum telescopes to simply F_{2n-1}.

Part vi: There is a beautiful way to explain this mysterious sum. Consider a row of $n+1$ circles with a second row of n circles centered beneath, and look only at paths whose first step is down/right. The remainder of such a path is equivalent to a path described in the Facts section (just upside down), so there are F_{2n} such paths in total. For any path through these $2n+1$ circles focus on the moves to the right along the top row.



There are n positions in which a right move could occur; a particular path will occupy some subset of these positions, creating an ordered partition of n. For instance, the path above corresponds to the ordered partition $2 + (n - 3) + 1$ of n. Therefore we group paths according to the partition of n arising from the right moves along the top row.

We claim that the number of paths in any such group is given by the product described in the problem statement. Suppose there is a string of a "blanks" before a path makes a right move in the upper row. This is achieved by starting with a down/right move, then repeatedly choosing $(a-1$ times) between a right move along the bottom row or an up/right, down/right pair. This can be done in 2^{a-1} ways. The same reasoning applies to each string of blanks in the top row. But blanks correspond to the odd-numbered positions within an ordered partition, confirming our claim. Therefore the sum gives the total number of paths, i.e. F_{2n}.

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