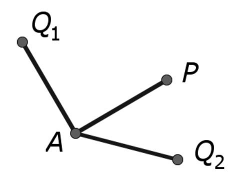
March 2010

The Mandelbrot Team Play

Round Three Test

Time Limit: 60 minutes

Facts: A *rotation* of the plane with center A through an angle θ leaves the center A fixed and carries any other point P to the point Q for which $QA = PA$ and $m\angle PAQ = \theta$. We call Q the *image* of P under the rotation. Here we adopt the usual convention that a positive angle corresponds to a counter-clockwise rotation, while a negative angle indicates a clockwise rotation. Thus in the diagram a $90°$ rotation about A carries P to Q_1, while a $-45°$ rotation carries P to Q_2.



Formulas: If a point P has coordinates $P(x, y)$, then the image Q upon rotating P by $90°$ about the origin has coordinates $Q(-y, x)$. More generally, rotating P by $90°$ about the center $A(a, b)$ results in an image with coordinates $Q(a + b - y, b - a + x)$.

Problems

Part i: (4 points) Label the points $A_1(0, 0)$, $A_2(4, 0)$ and $A_3(0, 3)$ in the coordinate plane. Plot the images Q_1, Q_2 and Q_3 when point $P(2, 1)$ is rotated $90°$ about points A_1, A_2 and A_3, respectively. Repeat this exercise when P has coordinates $(4, 3)$ or $(5, -2)$. In each case, how does $\triangle Q_1Q_2Q_3$ compare to $\triangle A_1A_2A_3$? Be as precise as possible, and include diagrams.

Part ii: (5 points) Let P and Q be arbitrary points in the plane. Show that the image obtained by rotating P by $90°$ about A is the same as rotating A by $-45°$ about P and then scaling outwards from P. Use this to explain your observations from the previous part.

Part iii: (5 points) Let A_1, A_2, A_3 and A_4 be the vertices of a quadrilateral, and let P be any point in the plane. If Q_1, Q_2, Q_3 and Q_4 are the images when P is rotated by $90°$ about A_1 to A_4 then find, with proof, the ratio $\mathit{area}(Q_1Q_2Q_3Q_4)/\mathit{area}(A_1A_2A_3A_4)$.

Part iv: (4 points) There is a unique point P with the property that rotating P by $90°$ about $A_1(3, 0)$, rotating the image by $90°$ about $A_2(-1, 4)$, then rotating that result by $90°$ about $A_3(-2, -1)$ brings us back to point P again. Find the coordinates of this point P.

Part v: (5 points) Suppose P_1, P_2, P_3 and P_4 have the property that rotating P_1 by $90°$ about P_2, rotating its image by $90°$ about P_3, then rotating that result by $90°$ about P_4 brings us back to P_1. Prove that rotating P_2 by $90°$ about P_3, P_4, and then P_1 returns us to P_2.

Part vi: (5 points) In $\triangle ABC$ let B_1 and B_2 be the points one-third and two-thirds of the way from A to B along $\overline{AB}$, and similarly define C_1 and C_2 along $\overline{AC}$. Rotate the centroid G by $90°$ about B_1 to obtain P, rotate G by $-90°$ about C_1 to obtain Q, and rotate C_2 by $90°$ about G to obtain R. Prove that $APRQ$ is a square.

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