Answer Key	
1. 291	4. 127
2. $\log_2 3$	5. 9.6
3. 49	6. 43
	7. 209

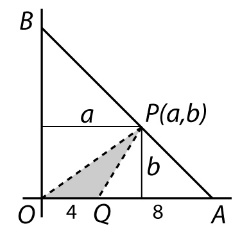
1. The three numbers—Jerry's original number and the two nearest multiples of five—are all about the same size. So as a first approximation we compute $\frac{1}{3}(876) = 292$. This suggests that the multiples of five used were 290 and 295, while Jerry's number was somewhere inbetween. Solving $290 + x + 295 = 876$ gives $x = \mathbf{291}$ as his original number.

2. By the definition of logarithms we have $\log_2 8 = 3$ since $2^3 = 8$. Therefore $\log_2 9 > 3$. But $\log_2 9 = 2\log_2 3$, hence $\log_2 3 > 1.5$. On the other hand, $\log_3 25 < \log_3 27 = 3$, so $\log_3 5 < 1.5$. Similarly, we find that $\log_5 11 < 1.5$ as well. To summarize, $\mathbf{\log_2 3}$ is the only value that is greater than 1.5, so it must be the largest.

3. We begin by observing that Abigail's number cannot be less than 40. For if it were, then according to the clues her number would have to be a perfect square and also one less than a multiple of 3. But the only perfect squares in this range are 1, 4, 9, 16, 25 and 36, and none of these are one less than a multiple of 3. Similarly, we claim that Abigail's number cannot be larger than 60, for if it were then by the clues it would be both a multiple of 7 and a prime, which is clearly impossible. Therefore we deduce that her number is somwhere from 40 to 60, inclusive, which means that it must be divisible by 7 and also a perfect square, by the clues. The only such number in this range is **49**.

4. We will first count how many positive integers from 1 to 1999 have the desired property. Note that it is equivalent to count how many multiples of 3 from 3 to 5997 have all even digits. Consider how we might build such a number: the units digit must be 0, 2, 4, 6 or 8, giving us five choices. Similarly we have five choices for the tens digit and five choices for the hundreds digit. The thousands digit must be either 0, 2 or 4; furthermore, the sum of the digits must be a multiple of 3 so that the entire number is divisible by 3. Conveniently, exactly one of the three possible thousands digits (0, 2 or 4) will produce a digit sum that is a multiple of 3. (Why?) Hence there are appear to be $5 \cdot 5 \cdot 5 = 125$ multiples of 3 having all even digits. However, we shouldn't include 0000 in our count, so there are actually 124 such numbers. Finally, we easily check that of the remaining numbers from 2000 to 2009, only 2000, 2002 and 2008 give all even digits when tripled, bringing the total to **127**.

5. We label points O, A, B, P and Q as shown in the diagram below. Both the x and y-intercepts are 12 for the line $x+y = 12$. In other words, $OA = 12$ and $OB = 12$, meaning that $\triangle OAB$ is a right isosceles triangle with $m\angle OAB = m\angle OBA = 45°$. But we also know that $m\angle OPB = m\angle QPA$, since the angle of incidence is equal to the angle of reflection when a laser beam reflects off a mirror. Therefore we may conclude that $\triangle OPB \sim \triangle QPA$ since these triangles have two pairs of angles that are equal. We know that $OB = 12$ and $QA = 12 - 4 = 8$, so the ratio of similarity is $12/8 = 3/2$. The altitudes of these triangles from point $P(a, b)$ have lengths a and b, which must also be in the same ratio. Thus $a/b = 3/2$ and $a + b = 12$ (since P lies on the line $x + y = 12$), which quickly gives $a = 7.2$ and $b = 4.8$. Finally, the area of $\triangle OPQ$ is given by



$$area(OPQ) = \tfrac{1}{2}(base)(height) = \tfrac{1}{2}(4)(4.8) = \mathbf{9.6}.$$

6. Pulling out as many factors of ξ as possible from the given expression leaves

$$\xi^{21}(2+\xi)(2+\xi^2)(2+\xi^3)(2+\xi^4)(2+\xi^5)(2+\xi^6).$$

The initial terms reduces to $(\xi^7)^3 = 1^3 = 1$. To handle the remaining product efficiently we note that the seventh roots of unity are exactly

1, ξ, ξ^2, ξ^3, ξ^4, ξ^5 and ξ^6. (Raising any of these to the seventh power gives 1.) Therefore these are the roots of the polynomial $x^7 - 1$, which means that

$$x^7 - 1 = (x-1)(x-\xi)(x-\xi^2)(x-\xi^3)(x-\xi^4)(x-\xi^5)(x-\xi^6).$$

Substituting in $x = -2$ and negating both sides shows that the above expression is equal to $(2^7+1)/(2+1) = \mathbf{43}$.

7. Suppose that the two markers simultaneously occupy the square in row $2m+1$ and column $2n+1$, for some $0 \le m \le 804$ and $0 \le n \le 1004$. Since the row and column are both odd-numbered, the markers will be moving to the right and down. The number of squares that the red marker traversed before reaching this square is $2009(2m)+2n$, while the green marker covered $(1609)(2n)+2m$ squares. These two expressions must be equal in order for the markers to reach the square at the same time, which means that $2008m = 1608n$, or $251m = 201n$. Solutions will be of the form $m = 201t$, $n = 251t$ for the five integers $0 \le t \le 4$.

If the markers instead meet in column $2n$ for $1 \le n \le 1004$, then the green marker will be moving up rather than down, so the number of squares traversed by each marker will be $2009(2m) + (2n-1)$ and $1609(2n-1) + (1608-2m)$. In this case our equation takes the form $2010m = 1608n$, or $5m = 4n$. Solutions will have the form $m = 4t$, $n = 5t$ for the 200 integers $1 \le t \le 200$. We leave the details for the final two cases to the reader, involving an even row and odd column or an even row and even column. There are 0 solutions and 4 solutions, respectively, in these cases. This brings the grand total to $5 + 200 + 0 + 4 = \mathbf{209}$ squares that the markers occupy at the same time.

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⋆ National Level ⋆

The Mandelbrot Competition

Round One Solutions

Greater Testing Concepts
PO Box 760
Potsdam, NY 13676

The Mandelbrot Competition
www.mandelbrot.org
info@mandelbrot.org