Answer Key	
1. 8	4. 74°
2. −5	5. 15
3. 468	6. 10
	7. 1509

1. We may put A and B either in the group with C or in the group with D. In the former case, there is only one person out of E, F, G, and H remaining to include in the group with C, leading to four ways to split up the people. Placing A and B with D instead again leads to four possibilities, for a total of **8** ways to form the groups.

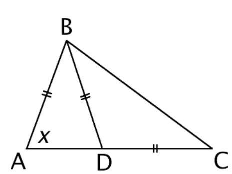
2. We can immediately deduce that $5 + \frac{4}{3+x}$ must equal 3, because 6 divided by this quantity gives 2. Therefore $\frac{4}{3+x} = -2$, which gives $3 + x = -2$. Finally, we discover that $x = -5$.

3. Since $2N$ is a three-digit number, N must be less than 500. Working backwards, we list the arithmetic numbers in this range, finding

$$486, 480, 471, 468, 465, 462, \ldots$$

Doubling each candidate in turn gives 972, 960, 942, 936, 930, 924, etc. The fourth number in this list is also arithmetic, so our answer is **468**.

4. It will be helpful to label $m\angle BAC = x$, as shown in the diagram at left.



Since $\triangle BAD$ is isosceles, this means that $m\angle BDA = x$ also, so $m\angle BDC = 180 - x$. Therefore the remaining two angles of $\triangle BDC$ must add up to x, which means that each angle is $\frac{1}{2}x$, since $\triangle BDC$ is isosceles as well. In summary, the angles of $\triangle ABC$ measure 69°, x, and $\frac{1}{2}x$, so

$$69^\circ + x + \tfrac{1}{2}x = 180^\circ \qquad \Longrightarrow \qquad \frac{3x}{2} = 111^\circ \qquad \Longrightarrow \qquad x = \mathbf{74}^\circ.$$

5. Let m be the number of marbles in the jar, and say that n of them are orange. If we draw a marble at random from the jar, the probability that it is orange is $\frac{n}{m}$. There are now $n-1$ orange marbles left among $m-1$ marbles, so the probability that the second draw is also orange is $\frac{n-1}{m-1}$. Thus we need

$$\frac{n}{m} \cdot \frac{n-1}{m-1} = \frac{1}{5} \qquad \Longrightarrow \qquad 5n(n-1) = m(m-1).$$

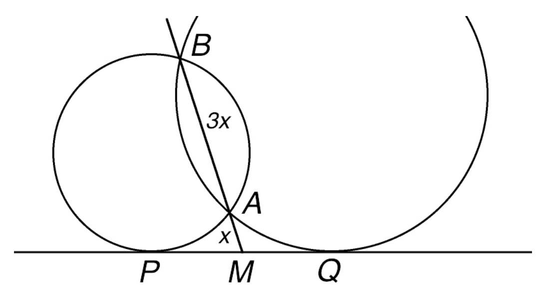
In other words, we need two products of consecutive integers, one of which is five times the other. Experimenting with various small values of n reveals that $5(6 \cdot 7) = 14 \cdot 15$, so we take $n = 7$ and $m = \mathbf{15}$. (There is a slightly involved procedure for finding *all* solutions to equations of this sort known as the theory of Pell equations.)

6. A quadratic equation has repeated roots exactly when its discriminant is zero. Writing the given equation as $x^2 + (z^5 + 2)x + (z^5 - 5)$ we see that it has the form $ax^2 + bx + c$ with $a = 1$, $b = z^5 + 2$, and $c = z^5 - 5$. The discriminant is

$$\begin{aligned} b^2 - 4ac &= (z^5+2)^2 - 4(z^5-5) \\ &= z^{10} + 4z^5 + 4 - 4z^5 + 20 \\ &= z^{10} + 24. \end{aligned}$$

Thus we need values of z for which $z^{10} = -24$. There are precisely **10** such values. (One can even list them all using roots of unity and $\sqrt[10]{24}$.)

7. Draw the line through A and B, meeting the x-axis at M as shown below. Since the vertical distance from A to B is three times the vertical



distance from the x-axis to A, using their y-coordinates, it follows that $AB = 3(MA)$ via similar triangles. So we label $MA = x$ and $AB = 3x$, giving $MB = 4x$. Now by Power of a Point (also sometimes called the Tangent-Secant Theorem in this setting) we know that

$$(MP)^2 = (MA)(MB) = 4x^2 \qquad \Longrightarrow \qquad MP = 2x.$$

In the same manner we deduce that $MQ = 2x$ also. Since $PQ = 2012$ we find that $MP = MQ = 1006$, and therefore $x = 503$. Finally this means that $AB = 3x = \mathbf{1509}$.

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⋆ Regional Level ⋆

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Round Five Solutions

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