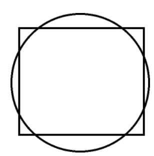
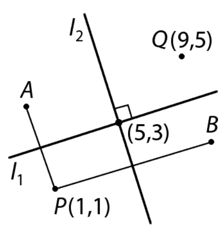
Answer Key	
1. 25	4. $4\sqrt{5}$
2. neither	5. 4/3
3. 10	6. 26112
	7. $15 - i$

1. Apparently the people in front of and behind Sonia account for a total of $32\% + 64\% = 96\%$ of the people in the line. Hence Sonia represents $4\% = 1/25$ of the people, so we deduce that there are **25** people in line.

2. It is in fact true that a line with negative slope must pass through quadrant *II*. However, a line such as $y = -3$ with *zero* slope does not have to pass through quadrant *II*. Hence the slope m is not necessarily positive. In the same manner, the line $y = x$ has a y-intercept of 0 (which is not negative) but does not pass through quadrant *II*. Therefore **neither** of the given statements must be true.

3. To increase the number of regions formed by the circle and rectangle we should increase the number of points at which they intersect. The circle may intersect each side of the rectangle at most twice, so we can obtain a maximum of **10** regions using a circle and a rectangle, as shown.





4. Observe that line l_2 has an angle of inclination exactly $90°$ more than line l_1, which implies that l_2 is perpendicular to l_1, as suggested by the diagram at right. But segment $\overline{PA}$ is also perpendicular to l_1, since P reflects across line l_1 to A. Therefore $\overline{PA} \parallel l_2$, and similarly $\overline{PB} \parallel l_1$. The upshot is that P, A, and B are three vertices of a rectangle. Reflecting $P(1,1)$ over the point of intersection at $(5,3)$ gives the fourth vertex at $(9,5)$, which we label Q. Because the diagonals of a rectangle are congruent, $AB = PQ = \sqrt{8^2+4^2} = \mathbf{4\sqrt{5}}$.

5. It stands to reason that in order to make a and d small, we should choose large values for b and c, then take $ab = 1$, $bc = 9$ and $cd = 1$. Letting $a = \frac{1}{3}$, $b = 3$, $c = 3$ and $d = \frac{1}{3}$ yields $a + 4d = \frac{5}{3}$, but it is possible to do better. Discovering the optimal value requires the use of the AM-GM inequality, which states that $\frac{1}{2}(x+y) \geq \sqrt{xy}$ for positive real numbers x and y. Using $x = a$ and $y = 4d$ yields

$$\frac{a+4d}{2} \geq \sqrt{4ad} = 2\sqrt{\frac{ab \cdot cd}{bc}} \geq 2\sqrt{\frac{1 \cdot 1}{9}} = \frac{2}{3}.$$

Hence $a + 4d \geq \mathbf{\frac{4}{3}}$, which can be achieved by taking $a = \frac{2}{3}$, $b = \frac{3}{2}$, $c = 6$ and $d = \frac{1}{6}$.

6. One of the most efficient means of performing a count such as this is to employ the Principle of Inclusion-Exclusion (PIE). We begin by observing that there are $8! = 40320$ ways for all eight students to be seated in any manner. However, in some cases Casey and Stacey will be seated next to one another, so we must subtract off these seating arrangements. This occurs $8 \cdot 6!$ times, since Casey could sit anywhere, then Stacey would be seated next to him, followed by the other six students sitting in any manner. The same is true for the other two pairs of estranged students, so we subtract off $3(8 \cdot 6!) = 17280$ seatings.

But now some cases have been subtracted off one too many times; namely, those arrangements in which two pairs of estranged students sit side by side. This occurs in $3(8 \cdot 6 \cdot 4!) = 3456$ cases (why?), so we add this amount back to our count. Finally, a few possibilities were added back once too many times—the situations in which all three pairs of estranged students are next to one another, which comes to $8 \cdot 6 \cdot 4 \cdot 2 = 384$ cases. Our grand total is $40320 - 17280 + 3456 - 384 = \mathbf{26112}$.

7. We write the terms of the geometric sequence as $z_1 = \alpha$, $z_2 = \alpha\beta$, and $z_3 = \alpha\beta^2$, where α and β are complex numbers. Based on the information we have on the average of these numbers and the average of their squares we know that

$$\alpha + \alpha\beta + \alpha\beta^2 = 30, \qquad \alpha^2 + \alpha^2\beta^2 + \alpha^2\beta^4 = 60i.$$

It turns out that solving for either α or β directly leads to an algebraic morass. The key to discovering the value of z_2 is to realize that the second expression factors as

$$\alpha^2 + \alpha^2\beta^2 + \alpha^2\beta^4 = (\alpha + \alpha\beta + \alpha\beta^2)(\alpha - \alpha\beta + \alpha\beta^2).$$

Thus dividing the second equation by the first yields $\alpha - \alpha\beta + \alpha\beta^2 = 2i$. Subtracting this from the first equation gives $2\alpha\beta = 30 - 2i$, hence the middle term is $z_2 = \alpha\beta = \mathbf{15 - i}$.

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