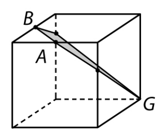
Answer Key	
1. 5	4. $\frac{2}{5}$, $\frac{4}{9}$
2. B	5. 19
3. \$26.50	6. 22
	7. $25/4^*$

*See below for an alternate acceptable answer.

1. The intersection of the plane with the cube is pictured at right as the shaded polygon. The polygon has sides along the top, front, left, right, and back sides. It now becomes clear the this polygon is a pentagon, having **5** sides.



2. For sake of comparison, let's suppose that Saelig paints 100 square feet in 60 minutes. (This is rather slow painting, but perhaps he has a small paintbrush.) Gabby, who paints 30% more area in the same amount of time, can cover $(100)(1.30) = 130$ square feet in 60 minutes. However, she spends 30% less time at the job, meaning she only spends $(60)(0.70) = 42$ minutes painting for each hour that Saelig works. Hence she only covers $(130)(0.70) = 91$ square feet in this time, whereas Saelig covers 100 square feet. Therefore **B** Saelig paints more.

3. Imagine that Marge only purchases the first type of bag; then she will need 7 bags in order to have enough apples and bananas. If she instead buys 6 bags then she must purchase 1 bag of the second type to have enough fruit. Continuing in this manner, we find the following possibilities, where each pair of numbers indicates the number of bags of the first and second type that must be bought.

$$(7,0) \quad (6,1) \quad (5,1) \quad (4,2) \quad (3,2) \quad (2,3) \quad (1,3) \quad (0,4)$$

Checking the price for each possibility reveals that the $(1,3)$ option is the cheapest, costing $1(\$4.00) + 3(\$7.50) = \$\mathbf{26.50}$.

4. Applying the process to the fraction $\frac{m}{n}$ twice in a row gives

$$\frac{m}{n} \to \frac{n-m+1}{n+m+2} \to \frac{(n+m+2)-(n-m+1)+1}{(n+m+2)+(n-m+1)+2} = \frac{2m+2}{2n+5}.$$

To obtain the original fraction again we must have

$$\frac{m}{n} = \frac{2m+2}{2n+5} \implies 2mn + 5m = 2mn + 2n,$$

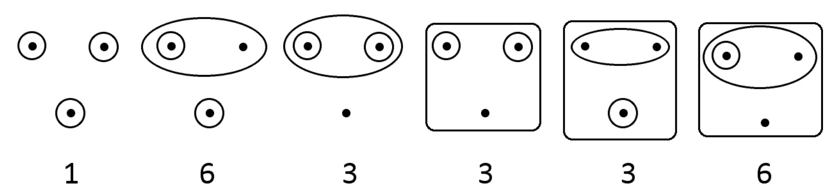
or $5m = 2n$. Since we want $\frac{m}{n}$ in lowest terms we take $m = 2$, $n = 5$, giving $\frac{2}{5}$. Applying our process repeatedly gives $\frac{2}{5} \to \frac{4}{9} \to \frac{2}{5} \to \frac{4}{9} \to \cdots$. Thus the two answers are $\mathbf{\frac{2}{5}}$ and $\mathbf{\frac{4}{9}}$.

5. Label the length and width of the rectangular pen as x and y. Then we are told that $x+2y = 20$ and that $(\frac{1}{2}x)^2 + y^2 = 9^2$. Rather than solve for x and y, we instead save time by combining the given equations in a manner that will immediately give the value of xy, which is the area we seek. Squaring the first equation and multiplying the second equation by 4 yields

$$x^2 + 4xy + 4y^2 = 400, \qquad x^2 + 4y^2 = 324.$$

Subracting the latter from the former gives $4xy = 76$, hence the area is $xy = \mathbf{19}$.

6. There are essentially six different types of diagrams involving three loops which encircle different sets of dots as described in the problem.



Furthermore, each diagram type shown above can be drawn (usually) in more than one way. Thus to draw the second type of diagram we choose a pair of dots to enclose (3 ways), then choose one dot from the pair to circle with a smaller loop (2 ways), then circle the last dot (1 way), for a total of 6 ways. The number of ways is indicated beneath each type, for a total of $1+6+3+3+3+6 = \mathbf{22}$ ways.

7. Let v represent the value of this infinite series. Then we have

$$\begin{aligned} v &= \alpha^2 + 2\alpha^5 + 3\alpha^8 + 4\alpha^{11} + \cdots \\ \alpha^3 v &= \qquad\ \ \alpha^5 + 2\alpha^8 + 3\alpha^{11} + \cdots \\ \Longrightarrow \quad (1-\alpha^3)v &= \alpha^2 + \alpha^5 + \alpha^8 + \alpha^{11} + \cdots, \end{aligned}$$

where we multiply the top equation by α^3 to obtain the second line, then subtract it from the first equation to reach the final line. We now have a geometric series with first term α^2 and common ratio α^3, hence

$$(1-\alpha^3)v = \frac{\alpha^2}{1-\alpha^3} \qquad \Longrightarrow \qquad v = \frac{\alpha^2}{(1-\alpha^3)^2}.$$

(The reader should confirm that $0 < \alpha < 1$, ensuring that this geometric series actually converges.) But we know that $1 - \alpha^3 = \frac{2}{5}\alpha$, since α satisfies the equation $x^3 + \frac{2}{5}x - 1 = 0$. Hence

$$v = \frac{\alpha^2}{(\frac{2}{5}\alpha)^2} = \mathbf{\frac{25}{4}}.$$

The answers $6\frac{1}{4}$ or 6.25 are also acceptable answers.

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⋆ Regional Level ⋆

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Round Five Solutions

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