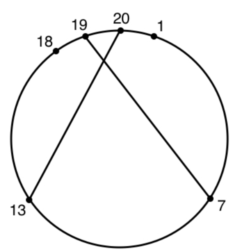
Answer Key	
1. 7	4. 4000
2. -3	5. $(7,24)$ $(8,15)$ $(9,12)$
3. $63°$	6. $1+\sqrt{3}$
	7. 105

1. When we add up the chips along each line (two horizontal and two vertical), we obtain a total of $(4)(5) = 20$ chips. Notice that each chip on a corner gets counted twice in this process, while the remaining chips are counted once. Or put another way, all 13 chips are counted once, then the chips on the corners are counted once more to bring the total up to 20, so clearly there must be **7** chips on the corners.

2. We see that $x = 1$ is a solution since $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$. To find the other solution we could either guess negative integers until we find one that works (probably the fastest approach!) or multiply through by the product $(x+1)(x+2)(x+5)$ to obtain

$$(x+2)(x+5) - (x+1)(x+5) = (x+1)(x+2).$$

Expanding yields $x^2 + 7x + 10 - (x^2 + 6x + 5) = x^2 + 3x + 2$, or just $x^2 + 2x - 3 = 0$, moving all terms to the right side. Factoring gives $(x-1)(x+3) = 0$, so the other solution is $x = \mathbf{-3}$.



3. A convenient means of keeping track of angles in a regular 20-sided polygon is via its circumscribed circle, since the twenty vertices divide the circumference into twenty congruent arcs, each measuring 18°. Lines $P_{20}P_{13}$ and $P_{19}P_7$ subtend arcs $P_{20}P_{19}$ and $P_{13}P_7$, which measure 18° and $6(18) = 108°$, respectively. According to a theorem on measures of angles formed by intersecting chords, the acute angle formed by these lines measures $\frac{1}{2}(18° + 108°) = \mathbf{63°}$.

4. Each of the congruent rectangles will be quite long and skinny, meaning that practically the entire perimeter will be taken up by the length, as opposed to the height. In other words, each rectangle has a width that is very close to $\frac{1}{2}(2013) = 1006.5$, which is the width of the original square. Therefore the square has a perimeter of about 4026, so we predict that its perimeter is **4000**, when rounded to the nearest hundred. (In case you were wondering, the actual perimeter is about 4022.)

5. We require the positive integers a, b and $\frac{1}{3}ab - a - b$ to satisfy the Pythagorean Theorem, which means that $a^2 + b^2 = (\frac{1}{3}ab - a - b)^2$. Expanding the right hand side gives

$$a^2 + b^2 = \tfrac{1}{9}a^2b^2 + a^2 + b^2 - \tfrac{2}{3}a^2b - \tfrac{2}{3}ab^2 + 2ab.$$

Cancelling the $a^2 + b^2$ on each side and dividing through by ab leads to

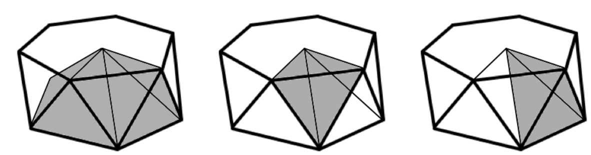
$$0 = \tfrac{1}{9}ab - \tfrac{2}{3}a - \tfrac{2}{3}b + 2.$$

Multiplying by 9, adding 18, and factoring yields

$$18 = ab - 6a - 6b + 36 = (a-6)(b-6).$$

We can now read off the pairs (a, b), since $(a - 6)$ and $(b - 6)$ must be positive integers multiplying to 18, such as $1 \cdot 18$, $2 \cdot 9$, or $3 \cdot 6$. The complete solution set is $(\mathbf{7}, \mathbf{24})$, $(\mathbf{8}, \mathbf{15})$, and $(\mathbf{9}, \mathbf{12})$.

6. The most likely strategy is to disect the solid into manageable pieces. Here is one such approach. Connect the center of the upper hexagon to all twelve vertices, forming a hexagonal pyramid along with a collection



of tetrahedra. The hexagonal pyramid shown on the left has base area of $6(\frac{1}{4}\sqrt{3})$ and height 1, and hence volume $\frac{1}{2}\sqrt{3}$. The tetrahedron in the middle has an equilateral base on the upper hexagon, thus volume $\frac{1}{3}(\frac{1}{4}\sqrt{3})(1)$. There are six of these tetrahedra, for a total volume of $\frac{1}{2}\sqrt{3}$

again. Finally, the tetrahedron on the right has perpendicular edges of length 1 on each of the upper and lower hexagons, separated by a height of 1. In this case the volume is simply $\frac{1}{6}lwh = \frac{1}{6}$. (This is not obvious, but we leave it to the reader to verify this fact.) Hence the total volume is $\frac{1}{2}\sqrt{3} + \frac{1}{2}\sqrt{3} + 6(\frac{1}{6}) = \mathbf{1 + \sqrt{3}}$.

7. How many coefficients of $(x^2+x+1)^{54}$ are divisible by 3? The answer turns out to be, almost all of them. The reason is that $54 = 2 \cdot 3^3$, combined with the fact that $(a+b)^3$ and a^3+b^3 differ by $3a^2b+3ab^2$, whose coefficients are multiples of 3.

More formally, we observe that $(a+b)^3 \equiv a^3+b^3 \bmod 3$. Applying this several times in succession, we find that

$$(a+b+c+d)^3 \equiv (a+b)^3 + (c+d)^3 \equiv a^3+b^3+c^3+d^3 \bmod 3.$$

To obtain the 54th power of (x^2+x+1) we first square it, then cube it three times, reducing mod 3 as we go. To begin,

$$(x^2+x+1)^2 = x^4+2x^3+3x^2+2x+1 \equiv x^4+2x^3+2x+1 \bmod 3.$$

Now cubing and using the fact just established gives

$$(x^4+2x^3+2x+1)^3 \equiv x^{12}+8x^9+8x^3+1 \bmod 3.$$

Replacing each 8 by 2 (since $8 \equiv 2 \bmod 3$) and repeating this process twice more, we eventually conclude that

$$(x^2+x+1)^{54} \equiv x^{108}+2x^{81}+2x^{27}+1 \bmod 3.$$

In other words, only four coefficients do *not* reduce to 0 mod 3, meaning that the remaining $109-4 = \mathbf{105}$ coefficients are multiples of 3.

December 2013

⋆ National Level ⋆

The Mandelbrot Competition

Round Two Solutions

Greater Testing Concepts
PO Box 760
Potsdam, NY 13676

The Mandelbrot Competition
www.mandelbrot.org
info@mandelbrot.org