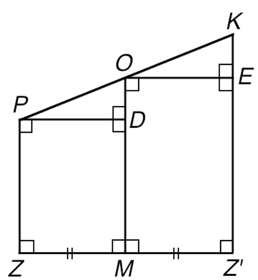
Team Play Solutions

Part i: Let M be the foot of the perpendicular from O to $\overline{AB}$. This creates right triangles OMZ and OMZ'. Furthermore, both triangles share leg $\overline{OM}$ and have congruent hypotenuses $\overline{OZ}$ and $\overline{OZ'}$. (The latter occurs because $\overline{OZ}$ and $\overline{OZ'}$ are radii of the same circle.) By hypotenuse-leg congruence we conclude that $\triangle OMZ \cong \triangle OMZ'$, which gives $\overline{MZ} \cong \overline{MZ'}$, as desired. Alternately, one could finish by using the Pythagorean Theorem to see that

$$MZ = \sqrt{(OZ)^2 - (OM)^2} = \sqrt{(OZ')^2 - (OM)^2} = MZ'.$$

Part ii: Consider the lines through Z, M and Z' perpendicular to $\overline{AB}$. We know by construction that the first two of these lines pass through P and O. Let the third line (through Z') meet line OP at point K. We will show that $PO = OK$, meaning that K is the same as point Q, which will complete the proof.



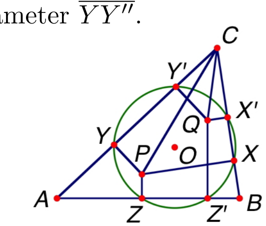
Draw segments through P and O perpendicular to $\overline{PZ}$ and $\overline{OM}$, as shown at right. This creates rectangles $PZMD$ and $OMZ'E$. Therefore $PD = ZM$ and $OE = MZ'$. Since we already know that $ZM = MZ'$, it follows that $PD = OE$. Furthermore, we have $\angle OPD \cong \angle KOE$ since $\overline{PD}$ and $\overline{OE}$ are parallel, and clearly angles $\angle PDO$ and $\angle OEK$ are both right angles. Therefore triangles $\triangle OPD$ and $\triangle KOE$ are congruent by ASA, giving $PO = OK$, as desired.

Part iii: Since angles $\angle PYC$ and $\angle PXC$ are both right angles, it follows that $CYPX$ is a cyclic quadrilateral as opposite angles are supplementary. Therefore

$$\angle ACP \cong \angle YCP \cong \angle YXP,$$

since $\angle YCP$ and $\angle YXP$ both intercept the same arc in circle $CYPX$. We deduce that $\angle BCQ \cong \angle X'Y'Q$ in exactly the same manner, using right angles $CY'Q$ and $CX'Q$ to get cyclic quadrilateral $CY'QX'$.

Part iv: There are probably a dozen different ways to establish that $\angle YXP \cong \angle X'Y'Q$, all of them involving an angle chase of some sort around the circle. Here is our favorite approach. First extend segment $\overline{XP}$ until it hits the circle again at X''. Note that since $\angle X'XX''$ is a right angle it follows that $\overline{X'X''}$ is a diameter. In the same way, extend $\overline{Y'Q}$ until it meets the circle at Y'', yielding diameter $\overline{YY''}$.



Now consider a rotation of the circle about its center O by $180°$. Since $\overline{X''X'}$ and $\overline{YY''}$ are diameters we know that points X'' and Y rotate onto points X' and Y''. Therefore arcs $X''Y$ and $X'Y''$ are congruent. But $\angle YXP$ and $\angle X'Y'Q$ are angles inscribed in the circle that intercept precisely these two arcs. By the Inscribed Angle Theorem they must be congruent, and you're done.

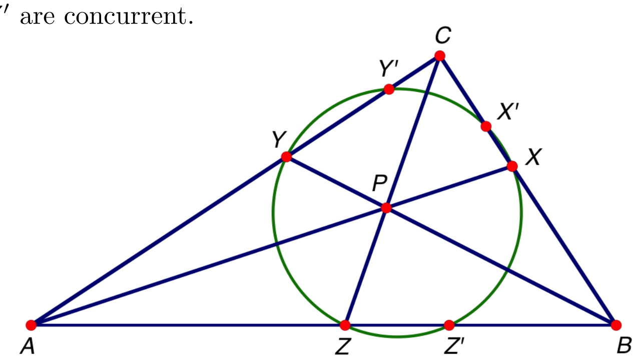
Part v: According to Ceva's Theorem we have

$$\frac{AY}{AZ} \cdot \frac{BZ}{BX} \cdot \frac{CX}{CY} = 1.$$

But Power of a Point with respect to A gives $(AZ)(AZ') = (AY)(AY')$, which may be rearranged to read $\frac{AY}{AZ} = \frac{AZ'}{AY'}$. Employing Power of a Point with respect to B and C also, then substituting into the above equality yields

$$\frac{AZ'}{AY'} \cdot \frac{BX'}{BZ'} \cdot \frac{CY'}{CX'} = 1.$$

Applying Ceva's Theorem once again we deduce that lines AX', BY' and CZ' are concurrent.



Part vi: To begin we observe that $(AY - AY')^2 = (CY - CY')^2$, since both quantities give the square of length YY'. In an analogous manner we compute each of $(XX')^2$ and $(ZZ')^2$ in two different ways and add all three equalities to obtain

$$(AY - AY')^2 + (BZ - BZ')^2 + (CX - CX')^2 = \\ (AZ - AZ')^2 + (BX - BX')^2 + (CY - CY')^2.$$

Power of a Point with respect to A gives $(AZ)(AZ') = (AY)(AY')$, and similarly with respect to B and C. Quadrupling each of these equalities and adding them into the mix has the effect of turning each cross term such as $-2(AY)(AY')$ into positive $2(AY)(AY')$, so $(AY - AY')^2$ becomes $(AY + AY')^2$. In summary, we find that

$$(AY + AY')^2 + (BZ + BZ')^2 + (CX + CX')^2 = \\ (AZ + AZ')^2 + (BX + BX')^2 + (CY + CY')^2.$$

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Round Three Solutions

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