Answer Key			
1.	36	4.	$4\frac{1}{4}$
2.	$1 < x < 3$	5.	17
3.	67	6.	2800/6561
		7.	$ab + a$

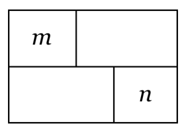
1. Recall that $0 + 1 + 2 + \cdots + 9 = 45$. Therefore if a two-digit number AB were the sum of the other eight digits, then adding $A + B$ would give the sum of all ten digits. So we would need

$$(10A + B) + A + B = 45 \qquad \Longrightarrow \qquad 11A + 2B = 45.$$

The only solution is $A = 3$, $B = 6$, so our two-digit number is **36**.

2. We begin by searching for values of x for which $5^x + 3 = 2^{2x+1}$. Remarkably, we find two integer solutions; namely, $x = 1$ and $x = 3$. Trying out $x = 2$ we discover that $5^2 + 3 = 28 < 2^5$. Hence $x = 2$ is part of the solution set, which therefore must be $\mathbf{1 < x < 3}$.

3. Let's organize our count according to the difference between the diagonally opposite numbers, labeled as m and n in the diagram.



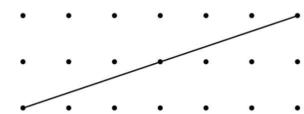
Case 1: If $n = m + 2$, such as the $m = 2$, $n = 4$ example given in the problem, then both remaining squares must be filled with the value $m + 1$. This leads to 3 possibilities. Similarly, when $n = m - 2$ there are again 3 possibilities.
Case 2: If $n = m + 1$, such as when $m = 1$, $n = 2$, then each remaining square must be labeled either m or $m + 1$, and all 4 possibilities work. There are four possible values for m, giving 16 more ways. Similarly, when $n = m - 1$ there are 16 ways.
Case 3: Finally, we might have $n = m$. There are several subcases to consider; the reader should find that when $m = 2$, 3, or 4 then there are 7 good labelings, while $m = 1$ or 5 leads to 4 good labelings.
All together we have found $3 + 3 + 16 + 16 + 3(7) + 2(4) =$ **67** ways.

4. There are a total of twelve shaded regions from left to right, separated by the vertical segments. Observe that turning the fifth region over and placing it on top of the twelfth region yields a perfect 1×1 square, obviously with area 1. The same can be done with the sixth and eleventh regions, the seventh and tenth regions, and the eighth and ninth regions, giving an area of 4 so far. Finally, the four remaining triangles have area

$$\begin{aligned} &\tfrac{1}{2}(\tfrac{1}{5})(\tfrac{1}{12}) + \tfrac{1}{2}(\tfrac{2}{5})(\tfrac{2}{12}) + \tfrac{1}{2}(\tfrac{3}{5})(\tfrac{3}{12}) + \tfrac{1}{2}(\tfrac{4}{5})(\tfrac{4}{12}) \\ &= \tfrac{1}{120}(1 + 4 + 9 + 16) = \tfrac{30}{120} = \tfrac{1}{4}, \end{aligned}$$

for an overall total area of $\mathbf{4\frac{1}{4}}$ or **4.25**.

5. Recall that Pick's Theorem states $\mathit{area} = I + \frac{1}{2}B - 1$, where I is the number of interior points and B is the number boundary points of the quadrilateral, including the vertices. We are given that there are eight interior points, so $I = 8$. The key to finishing is to notice that a lattice segment of length $2\sqrt{10}$ can only arise if it has slope $\frac{1}{3}$ (as shown at right), or slope $-\frac{1}{3}$, or slope ± 3. Regardless, such a segment involves three boundary points. In a similar fashion a segment of length $3\sqrt{13}$ has four boundary points, a segment of length $4\sqrt{2}$ has five boundary points, and a segment of length 11 has twelve boundary points. But each vertex is counted twice along the way, hence



$$B = 3 + 4 + 5 + 12 - 4 = 20, \quad \Longrightarrow \quad \mathit{area} = 8 + \tfrac{1}{2}(20) - 1 = \mathbf{17}.$$

6. It's quite helpful to realize that the parity of A is unchanged by adding in twice the area of overlap, giving a new value $\mathit{area}(R_1) + \mathit{area}(R_2)$. This sum will be odd if $\mathit{area}(R_1)$ is odd while $\mathit{area}(R_2)$ is even, or the other way around. But for $\mathit{area}(R_1)$ to be odd both the length and width must be odd; each of these occurs with probability

$$\frac{9 + 7 + 5 + 3 + 1}{9 + 8 + \cdots + 2 + 1} = \frac{25}{45} = \frac{5}{9},$$

so the probability that $\mathit{area}(R_1)$ will be odd is $\frac{25}{81}$. Therefore the probability of an even area is $1 - \frac{25}{81} = \frac{56}{81}$, meaning that the overall answer is given by

$$2 \cdot \frac{25}{81} \cdot \frac{56}{81} = \mathbf{\frac{2800}{6561}}.$$

7. The instructions described in this problem function in a manner analogous to lines of computer code. Following the directions, we multiply by $\frac{35}{2}$ a times, giving $3^b 5^a 7^a$. We next multiply by $\frac{11}{3}$ to get $3^{b-1}5^a 7^a 11$, followed immediately by $\frac{26}{77}$ and $\frac{11}{13}$, which as a pair effectively replace a 7 by a 2 in the prime factorization as long as an 11 is present. Thus doing this repeatedly brings us to $2^a 3^{b-1} 5^a 11$, at which point we multiply by $\frac{1}{11}$ instead. In summary, so far we have

$$2^a 3^b \quad \longrightarrow \quad 3^b 5^a 7^a \quad \longrightarrow \quad 2^a 3^{b-1} 5^a.$$

Observe that the 5s don't affect any steps in this process, but rather just accumulate. Hence the entire previous sequence of steps will play out again, bringing us to

$$2^a 3^b \quad \longrightarrow \quad 2^a 3^{b-1} 5^a \quad \longrightarrow \quad 2^a 3^{b-2} 5^{2a}.$$

Continuing in this manner we eventually reach $2^a 3^0 5^{ab}$, but this time after multiplying by $\frac{35}{2}$ a times we obtain $5^{ab+a}7^a$, so we can't multiply by $\frac{11}{3}$ as before. Instead, we reach the end game in which we multiply by $\frac{1}{7}$ until we are left with just 5^{ab+a}, so our answer is $\boldsymbol{ab+a}$.

Problem Credits

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1. Isaac	3. Dr.V	6. Isaac
2. Isaac	4. Oliver	7. Ryan

December 2018

⋆ National Level ⋆

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Round Three Solutions

Produced by
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