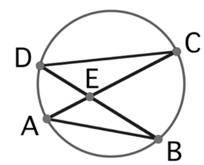
March 2014

The Mandelbrot Team Play

Round Three Test

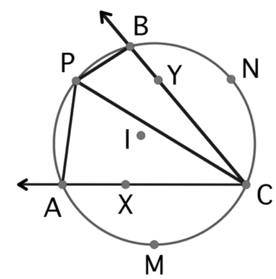
Time Limit: 60 minutes

Facts: The *Intercepted Arc Theorem* states that if points A, B and C are on a circle, then $m\angle CAB = \frac{1}{2}m\widehat{BC}$. In other words, an angle inscribed in a circle measures half its intercepted arc. An extension of this result applies to angles within a circle; in the diagram at right we have $m\angle BEC = \frac{1}{2}(m\widehat{AD} + m\widehat{BC})$.



Let W be a point on a given segment $\overline{UV}$ other than its midpoint. Then the set of points Z in the plane such that $ZU/ZV = WU/WV$ is a *circle of Apollonius*, whose center lies on line UV. (Only needed for the final part.)

Setup: Given points A, B and C on a circle, let P be any point in the interior of angle $\angle ACB$. Suppose the angle bisector of $\angle APC$ meets segment $\overline{AC}$ at X and crosses arc $\widehat{AC}$ at its midpoint M, while the angle bisector of $\angle BPC$ meets segment $\overline{BC}$ at Y and crosses $\widehat{BC}$ at its midpoint N. Below we will prove that line XY passes through I, the incenter of $\triangle ABC$, if and only if P lies on the circle.



Problems

In the first four parts we suppose that P is on the circle and prove that $\overline{XY}$ passes through I.

Part i: (4 points) To begin, show that triangles $\triangle MAX$ and $\triangle MPA$ are similar.

Part ii: (4 points) Next explain why $\angle MAI \cong \angle AIM$.

Part iii: (5 points) Combine the previous results to prove that $\triangle MIX \sim \triangle MPI$.

Part iv: (5 points) The same reasoning reveals that $\triangle NIY \sim \triangle NPI$. (You may use this fact without proof.) Now prove that $\angle XIA \cong \angle YIN$ to deduce that I lies on $\overline{XY}$.

We conclude by proving that if we move P off the circle then $\overline{XY}$ passes between I and C.

Part v: (5 points) At first we will keep P on the circle. Let ω_1 be the circle through points P and X whose center lies on line AC. In the same manner, let ω_2 be the circle through points P and Y whose center lies on line BC. Prove that ω_1 and ω_2 are tangent at P.

Part vi: (5 points) Finally, consider what happens to the positions of points X and Y as P varies along ω_1 to prove that if P is not on the circle, then $\overline{XY}$ passes between I and C.