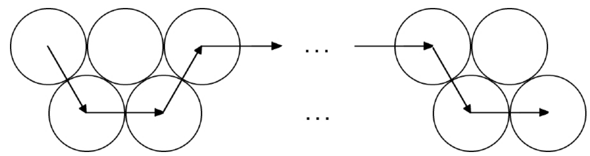
January 2012

The Mandelbrot Team Play

Round One Test

Time Limit: 60 minutes

Facts: Consider a row of n adjacent circles with a second row of n circles beneath and slightly to the right, just touching the first row, as shown below. A path through these circles consists of a sequence of steps from the upper left to lower right circle, always moving directly to the right, up/right, or down/right. One such path is illustrated in the diagram. There are a total of F_{2n} paths through the circles, where F_{2n} is a Fibonacci number, defined by $F_1 = 1$, $F_2 = 1$, and $F_{k+1} = F_k + F_{k-1}$ for $k \geq 2$.



Setup: An *ordered partition* of a positive integer n is a way of writing n as a sum of one or more positive integers, where the order of the sum matters. Thus there are four ordered partitions of 3, namely 3, $2+1$, $1+2$ and $1+1+1$. Now suppose that we turn each such sum into a product, yielding 3, $2 \cdot 1$, $1 \cdot 2$, and $1 \cdot 1 \cdot 1$. Adding together these four "ordered partition products" yields $3+2+2+1 = 8$. In the subsequent questions you will demonstrate the remarkable fact that this final sum is always a Fibonacci number.

Problems

Part i: (4 points) Write out all the ordered partitions of 4, change each sum to a product, then add together all the products. Demonstrate that the result is in fact a Fibonacci number.

Part ii: (4 points) How many paths are there through two rows of n circles if we are only allowed to move directly to the right or down/right? Next calculate the number of paths if the fourth step is up/right but all others are right or down/right. (Assume that $n \geq 4$.)

Part iii: (5 points) Sort all paths according to the location of the up/right steps. So all paths moving up/right at the second and fifth steps (but nowhere else) belong to one group, and so on. Describe how the groups that arise are related to ordered partitions.

Part iv: (5 points) Combine the ideas from the previous three parts to deduce that the sum of the ordered partition products of n is always a Fibonacci number, as claimed.

Part v: (5 points) Suppose we replace the final term in each ordered partition of n with a 1 before multiplying. Show that the sum of these products is also a Fibonacci number.

Part vi: (5 points) Given a sum keep only the first, third, fifth, ... terms. Replace each such term a by 2^{a-1} and multiply the results. Thus $3+5+1+2+4$ would become $2^2 2^0 2^3 = 32$. Prove that doing this to every ordered partition of n and adding the results gives F_{2n}.