Team Play Solutions

Part i: The table appears below, oriented horizontally to save space.

n	0	1	2	3	4	5	6	7	8	9	10	11
binary	0	1	10	11	100	101	110	111	1000	1001	1010	1011
$b(n)$	0	1	1	2	1	2	2	3	1	2	2	3

12	13	14	15	16	17	18	19	20	21
1100	1101	1110	1111	10000	10001	10010	10011	10100	10101
2	3	3	4	1	2	2	3	2	3

22	23	24	25	26	27	28	29	30	31
10110	10111	11000	11001	11010	11011	11100	11101	11110	11111
3	4	2	3	3	4	3	4	4	5

Carefully adding together the entries along the bottom row reveals that $b(0) + b(1) + b(2) + \cdots + b(31) = 80$.

Part ii: Observe that the binary representations for k and $31 - k$ are "complements" of one another, in the sense that changing the 0's to 1's and vice-versa in one binary number gives the other. This is the case because $31 = 11111_2$, so subtracting $31 - k$ in binary has the effect of replacing each 0 in the binary representation of k by a 1 and each 1 by a 0. (In the same way, subtracting $99999 - 45545$ gives 54454 in base ten; the 4's become 5's and vice versa.) So between them k and $31 - k$ have a total of five 1's in binary, hence $b(k) + b(31 - k) = 5$.

In the same way k and $(2^r - 1) - k$ for $k = 0, 1, \ldots, 2^r - 1$ are "binary complements," since $2^r - 1 = 11 \cdots 1_2$ with r 1's in its binary representation. Therefore we can group these 2^r numbers into 2^{r-1} pairs, with $b(k) + b(2^r - 1 - k) = r$ for each pair, giving an overall total of

$$b(0) + b(1) + b(2) + \cdots + b(2^r - 1) = r2^{r-1}.$$

Note that this formula agrees with the previous part when $r = 5$.

Part iii: We first realize that since $m+n < 2011$, each of m, n and $m+n$ has at most 11 digits in binary. Imagine performing the sum $m + n$ in binary; a sample computation is shown at right as an illustration. To the immediate right of each column which does *not* involve carrying a 1, we draw vertical bars, which split the sum into various blocks. Now observe that the rightmost column of each block reads 1–1–0 (vertically, of course), the leftmost block reads 0–0–1, and each middle column is either 1–0–0, 0–1–0 or 1–1–1. This is due to the way in which binary addition in performed.

$$\begin{array}{c cccc|cc|ccccc}
 & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & {\scriptstyle 1} & & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & \\
 & & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
+ & & & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
\hline
 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0
\end{array}$$

We now compare the number of 1's in the top two rows with those in the bottom row. Within any block each middle row has one more 1 above than below the line. The same is also true of the right and left columns combined. Therefore a block of c columns has $c - 1$ more 1's above than below the line. This means that a binary addition with 11 columns will have at most 10 more 1's above the line than below, proving that $b(m) + b(n) - b(m + n) < 11$. Incidentally, it is possible to actually achieve a difference of 10, for instance with $m = 1$ and $n = 1023$.

Part iv: We prove the claim by induction. To begin, $v(1) = 0$ while $1 - b(1) = 1 - 1 = 0$, so the two quantities agree for $n = 1$. In addition, $v(1) + v(2) = 0 + 1 = 1$ and $2 - b(2) = 2 - 1 = 1$, so the equality holds for $n = 2$ as well. Now suppose the statement is true for $n = k$. We wish to show that this implies that the statement is true for $n = k + 1$. So we are given that

$$v(1) + v(2) + \cdots + v(k) = k - b(k).$$

Adding $v(k + 1)$ to both sides of the first statement gives

$$v(1) + v(2) + \cdots + v(k) + v(k + 1) = k - b(k) + v(k + 1),$$

so once we show that $k - b(k) + v(k+1) = k+1 - b(k+1)$ we are done. This equality simplifies to just $b(k) - b(k + 1) = v(k + 1) - 1$. Suppose that k and $k + 1$ are written in binary as

$$k = 1 \cdots 0 \underbrace{11 \cdots 1}_{n\ 1's}, \qquad k + 1 = 1 \cdots 1 \underbrace{00 \cdots 0}_{n\ 0's},$$

for some $n \geq 0$. Then $b(k) - b(k+1) = n - 1$, while $v(k+1) = n$. Hence it is true that $b(k) - b(k + 1) = v(k + 1) - 1$, and you're done.

Part v: There are at least *two* ways to generalize $b(n)$ for base three: either let $b_3(n)$ count the number of nonzero digits in the base three representation for n, or else let $b_3(n)$ be the sum of those digits. It turns out that the latter more readily provides a base three analogue to the previous part. Some experimentation leads to the formula

$$v_3(1) + v_3(2) + \cdots + v_3(n) = \tfrac{1}{2}(n - b_3(n)),$$

where $v_3(n)$ counts the number of factors of 3 in the prime factorization of n. For instance, when $n = 11$ we have

$$v_3(1) + \cdots + v_3(12) = 0+0+1+0+0+1+0+0+2+0+0 = 4.$$

Meanwhile $11 = 102_3$ so $b_3(11) = 3$. As predicted, $\frac{1}{2}(11-3) = 4$. This statement can be proved similarly to the previous part.

Part vi: We opt for a combinatorial argument. The lefthand side of the given identity counts how many pairs of 1's there are in the numbers from $00\cdots0_2$ up to $11\cdots1_2$, the 2^r numbers that can be created with r binary digits. An alternate way to count this same quantity is to focus our attention on two digits (such as the last two) and figure out how many binary numbers include a pair of 1's in these positions. The answer is 2^{r-2}, since each of the remaining $r-2$ digits could be either a 0 or 1. Hence this particular pair of 1's is counted 2^{r-2} times in the sum. The same is true of all $\binom{r}{2}$ pairs of digits, giving the desired result.

Multiplying this identity by 2 and adding the identity discovered in part ii now reveals that

$$\begin{aligned} b(0)^2 + b(1)^2 + b(2)^2 + \cdots + b(2^r-1)^2 &= 2\binom{r}{2}2^{r-2} + r2^{r-1} \\ &= (r^2 - r)2^{r-2} + 2r2^{r-2} \\ &= (r^2 + r)2^{r-2}. \end{aligned}$$

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Round One Solutions

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