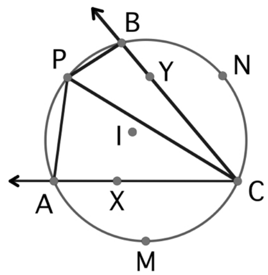
Team Play Solutions

We include the diagram below for easy reference.



Part i: Observe that we know the angle bisector of $\angle APC$ crosses $\widehat{AC}$ at its midpoint M, because $\angle APM \cong \angle MPC$ implies that $m\widehat{AM} = m\widehat{MC}$ by the Intercepted Arc Theorem. But this means that

$$m\angle MAX = \tfrac{1}{2}m\widehat{MC} = \tfrac{1}{2}m\widehat{AM} = m\angle MPA.$$

Thus triangles $\triangle MAX$ and $\triangle MPA$ have one pair of congruent angles. But they also share the angle at vertex M, hence they are similar by the angle-angle (AA) similarity criterion.

Part ii: To simplify notation, we write β for the common measure of arcs $\widehat{AM}$, $\widehat{MC}$ and let α be the measure of $\widehat{BN}$, $\widehat{NC}$. Notice that since I is the incenter it follows that line AI bisects $\angle BAC$, hence passes through the midpoint N of arc $\widehat{BC}$. Thus

$$m\angle MAI = \tfrac{1}{2}m\widehat{MN} = \tfrac{1}{2}(\alpha+\beta).$$

On the other hand, the angle formula from the Facts section gives

$$m\angle AIM = \tfrac{1}{2}(m\widehat{AM} + m\widehat{BN}) = \tfrac{1}{2}(\beta+\alpha).$$

Therefore $\angle MAI \cong \angle AIM$, as desired.

Part iii: The similar triangles in part i imply $MX/MA = MA/MP$. But from part ii we know $\triangle AIM$ is isosceles with $MA = MI$ because its base angles are congruent. Hence $MX/MI = MI/MP$ upon substituting, which shows that two pairs of corresponding sides are proportional in triangles $\triangle MIX$ and $\triangle MPI$. These triangles also have a common angle at vertex M, hence they are similar by the side-angle-side (SAS) similarity criterion.

Part iv: The means of linking the two sides of the diagram together is via angle $\angle MPN$. First note that $\angle MPN \cong \angle MAI$ since they both intercept arc $\widehat{MN}$, and we already know $\angle MAI \cong \angle AIM$. Therefore

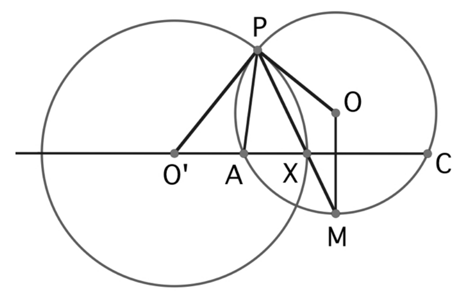
$$m\angle MPN = m\angle AIM = m\angle MIX + m\angle XIA.$$

But we also know that $\triangle MIX \sim \triangle MPI$ and $\triangle NIY \sim \triangle NPI$, hence

$$m\angle MPN = m\angle MPI + m\angle NPI = m\angle MIX + m\angle NIY.$$

Equating these two expressions yields $\angle XIA \cong \angle NIY$. Finally, since it is known that A, I and N are collinear, this forces X, I and Y to also be collinear, meaning that I lies on $\overline{XY}$.

Part v: We will show that ω_1 is orthogonal to the circle through A, B and C at point P. It will follow by the same reasoning that ω_2 is also orthogonal at P, which then proves that ω_1 and ω_2 are tangent to one another at P.



Let O be the center of the circumcircle of $\triangle ABC$, and let O' be the center of ω_1 on line AC. We wish to show that $m\angle OPO' = 90°$. But

$$m\angle O'PX = m\angle AXP = \tfrac{1}{2}(m\widehat{AP} + m\widehat{MC}) = \tfrac{1}{2}(m\widehat{AP} + m\widehat{AM})$$
$$= \tfrac{1}{2}m\widehat{PM} = \tfrac{1}{2}m\angle MOP = \tfrac{1}{2}(180° - 2m\angle OPX) = 90° - m\angle OPX.$$

Here we used the fact that triangles $\triangle O'PX$ and $\triangle OPM$ are isosceles, along with the fact that $m\widehat{PM}$ is equal to its central angle $m\angle POM$. Hence angles $\angle O'PX$ and $\angle OPX$ are complementary, giving the desired right angle $\angle O'PO$ and showing that the circles are orthogonal at P.

Part vi: Consider the set of all points Z for which $ZA/ZC = XA/XC$. By the Facts section this is a circle through X with center on line AC. But by the Angle Bisector Theorem we know that $PA/PC = XA/XC$, hence P is also on this circle, so we deduce this circle is ω_1. The upshot is that as P varies along ω_1, the ratio PA/PC does not change, so the angle bisector of $\angle APC$ continues to meet $\overline{AC}$ at X. However, P is now outside ω_2, since these circles are tangent, making the ratio PB/PC larger than before. (The reader should confirm that the ratio is larger rather than smaller.) By the Angle Bisector Theorem, this means that YB/YC will also increase, having the effect of moving Y closer to C. Since X stays put, we conclude that for positions of P other than on the circumcircle of $\triangle ABC$, it is the case that $\overline{XY}$ passes between I and C, as claimed.

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Round Three Solutions

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