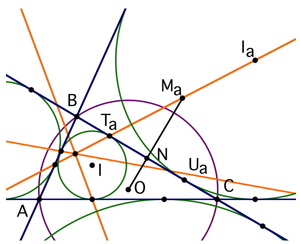
Team Play Solutions

Part i: We first observe that N is also the midpoint of side $\overline{BC}$, because

$$BN = BT_a + T_aN = CU_a + U_aN = CN,$$

using the fact that $BT_a = CU_a$. But $OB = OC$ as well, since they are both radii of the circle through A, B and C. Hence we can conclude that $\triangle ONB \cong \triangle ONC$, which implies that $\angle ONB$ and $\angle ONC$ are both right anges, since they are congruent and their measures sum to $180°$.



Part ii: We have just seen that line ON is perpendicular to side $\overline{BC}$. But lines IT_a and I_aU_a are also perpendicular to $\overline{BC}$, since a radius out to a point of tangency is perpendicular to the tangent line. Therefore ON, IT_a and I_aU_a are parallel lines. Furthermore, $NT_a = NU_a$, so these parallel lines are equally spaced. It follows that they cut off equal segments on any transversal, such as line I_aT_a, which shows that line ON meets segment I_aT_a at its midpoint M_a. (*N.B.* It is fine to prove this fact "from scratch," but equally spaced lines make the result more transparent.)

Part iii: Recall that $\triangle ONB \cong \triangle ONC$, so we know that angles $\angle BON$ and $\angle CON$ are congruent. Since O is the center of the circle through A, B and C, we deduce that arcs $\widehat{BQ}$ and $\widehat{CQ}$ are congruent. But this means that $\angle BAQ \cong \angle CAQ$, because these angles intercept congruent arcs, and an angle measures half the size of its intercepted arc. In other words, Q lies on the angle bisector of $\angle BAC$ as claimed. Since we already know that I and I_a are located on this angle bisector, all four points A, I, Q and I_a are situated on a single line.

Part iv: We will show that triangles $\triangle IT_aI_a$ and $\triangle QM_aI_a$ are similar. By the previous part I, Q and I_a are situated on a single line, so these triangles share a common angle at vertex I_a. Furthermore, sides IT_a and QM_a are parallel by part ii, so the remaining angles are also congruent since they form pairs of corresponding angles. Finally, we observe that $\triangle QM_aI_a$ is exactly half the size of $\triangle IT_aI_a$, since $M_aI_a = \frac{1}{2}T_aI_a$. Therefore $QM_a = \frac{1}{2}IT_a$ as well, which means $QM_a = \frac{1}{2}r$ since IT_a is a radius of the incircle.

Letting R be the radius of the circumcircle through A, B and C, we have $OM_a = R + \frac{1}{2}r$. By the same reasoning presented in these first four parts so far, we may conclude that $OM_b = R + \frac{1}{2}r$ and $OM_c = R + \frac{1}{2}r$ as well. But this means that the circle with center O and radius $R + \frac{1}{2}r$ passes through M_a, M_b and M_c. In other words, this circle is concentric with the circumcircle of $\triangle ABC$, which is pretty cool.

Part v: Let V be the point where line I_aT_a intersects line OI. (Ignore lines I_bT_b and I_cT_c for the time being.) Since lines IT_a and OM_a are parallel, we can immediately deduce that $\triangle VIT_a \sim \triangle VOM_a$. And since we know the lengths of IT_a and OM_a, we can also determine the ratio between corresponding sides of these triangles, namely

$$\frac{OV}{IV} = \frac{OM_a}{IT_a} = \frac{R + \frac{1}{2}r}{r} = \frac{R}{r} + \frac{1}{2}.$$

Again, applying the same sequence of steps for lines I_bT_b and I_cT_c, we would discover that these lines intersect line OI at a point creating the same ratio $\frac{R}{r} + \frac{1}{2}$. But there is a unique point V on line OI (on the other side of I from O) satisfying $\frac{OV}{IV} = \frac{R}{r} + \frac{1}{2}$, so lines I_bT_b and I_cT_c must also cross line OI at this same point. In summary, we conclude that all four of lines OI, I_aT_a, I_bT_b and I_cT_c are concurrent at point V.

Part vi: We exploit one last pair of similar triangles to deduce this neat relationship between R, r and r_a. Extend T_aO until it intersects line I_aU_a, say at point P. Then $\triangle OT_aM_a$ is similar to $\triangle PT_aI_a$ and is half as large, since M_a is the midpoint. It follows that $PI_a = 2(OM_a) = 2R + r$, by part iv. On the other hand, we find that

$$PI_a = PU_a + U_aI_a = 2(ON) + r_a,$$

using the fact that $PU_a = 2(ON)$ via $\triangle T_aON \sim \triangle TPU_a$. It remains to show that $ON = R\cos A$. But this follows immediately from right triangle $\triangle BON$ where $BO = R$, and

$$m\angle BON = m\widehat{BQ} = 2m\angle BAQ = m\angle A,$$

as shown in part iii, and we're done. Of course, it is possible (with some effort) to establish that $r_a + 2R\cos A = 2R + r$ using only standard formulas involving various lengths and angles associated with $\triangle ABC$, but we hope that you will agree that the geometric demonstration is much more direct and illuminating.

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Round Three Solutions

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