Team Play Topics

ROUND THREE

The first section of the Round Three Mandelbrot Team Play is reproduced below. A list of topics and practice problems are also provided to aid in preparation. Note that these problems are not meant to serve as a precise indicator of the problems that will appear on the contest. However, students who understand how to solve them should be able to make significantly more progress than they might have otherwise. So work hard on the problems, and good luck on the Team Play!

Facts: The *Theorem on Cyclic Quadrilaterals* states that if points C and D are situated on the same side of line AB then $\angle ACB \cong \angle ADB$ exactly when the four points lie on a single circle. Similarly, if C and D are on opposite sides of line AB, then angles $\angle ACB$ and $\angle ADB$ are supplementary if and only if the points lie on a circle. The *Power of a Point Theorem* asserts that if two lines through a given point P intersect a circle at pairs of points A, B and C, D then we have $(PA)(PB) = (PC)(PD)$.

TOPICS: Circle facts, cyclic quadrilaterals, Power of a Point Theorem, Ceva's Theorem

Practice Problems

The first four problems below demonstrate that the altitudes of a triangle are concurrent. The strategy for the proof is adapted from *Episodes in Nineteenth and Twentieth Century Euclidean Geometry,* by Ross Honsberger.

1. In triangle ABC, let M be the midpoint of $\overline{BC}$ and let O be the center of the circle through A, B and C. Explain why $\overline{OM} \perp \overline{BC}$.

2. Use Ceva's Theorem to show that the medians of $\triangle ABC$—the segments from each vertex to the midpoint of the opposite side—are all concurrent. (Recall that the centroid G through which the medians all pass is two-thirds of the way from each vertex to the opposite midpoint.)

3. Construct point H on line OG so that $HG = 2(OG)$ and G is between H and O. Prove that $\triangle AHG \sim \triangle MOG$.

4. Finally, explain why the altitude from A to $\overline{BC}$ passes through point H. How does this show that the altitudes are concurrent?

5. Let X, Y and Z be the feet of the altitudes from A, B and C to the opposites sides of the triangle. Show that $BXHZ$ is cyclic quadrilateral, and deduce that $\angle HXZ \cong \angle HBZ$.

6. Prove that altitude $\overline{AX}$ is the angle bisector of $\angle YXZ$.

Hints and answers on the next page. $\Longrightarrow$

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Hints and Answers

1. Show that $\triangle OMB \cong \triangle OMC$ by SSS. Conclude that since angles $\angle OMB$ and $\angle OMC$ are both congruent and supplementary, each must be a right angle.

2. Let the midpoints of $\overline{AB}$, $\overline{AC}$ and $\overline{BC}$ be K, L and M. Ceva's Theorem states that the medians are concurrent if and only if $\frac{AK}{KB}\frac{BM}{MC}\frac{CL}{LA} = 1$. But since $AK = KB$ and similarly this expression clearly reduces to 1.

3. We have vertical angles $\angle OGM$ and $\angle HGA$. Furthermore, we know that $\frac{OG}{GH} = \frac{MG}{GA} = \frac{1}{2}$. Now use SAS similarity.

4. Here is one way of looking at it. Triangle AGH is obtained from triangle MGO by spinning it 180° about G and scaling up by a factor of 2. This process takes line OM and yields a parallel line AH. Since $\overline{OM} \perp \overline{BC}$, we deduce that $\overline{AH} \perp \overline{BC}$ also, meaning that line AH is the altitude.

5. We know that angles $\angle HXB$ and $\angle HZB$ are both right angles, hence are supplementary. So the Theorem on Cyclic Quadrilaterals implies that these four points lie on a single circle. By this same theorem, we know that $\angle HXZ \cong \angle HBZ$, since these angles are inscribed in the same arc.

6. By the above reasoning, we may also deduce that $\angle HXY \cong \angle HCY$. So we need only show that $\angle HBZ \cong \angle HCY$. But this follows at once from the fact that both angles are complementary to $\angle BAC$.