Answer Key	
1. $2\frac{1}{2}$	4. $(3,-2,2)$
2. B	5. 61
3. 6	6. $\frac{1}{4}$
	7. $2\sqrt{5}$

1. The tortoise walks 1 foot every 4 seconds, so he will need 2000 seconds to finish the 500 foot course. The hare will only take $500/10 = 50$ seconds to cover the same distance, but we must add in $60 \cdot 30 = 1800$ seconds for her nap. She still wins though, by $2000 - 1850 = 150$ seconds, which translates to $\mathbf{2\frac{1}{2}}$ minutes.

2. One approach is to employ the quadratic formula to compute the positive roots of each function, then compare them directly. One should find that the positive root of the second quadratic is the largest. Or note that the graph of each function is a parabola. Decreasing the constant term lowers the parabola, which has the effect of spacing the roots out, so the positive root becomes larger. Similarly, decreasing the linear coefficient (from $5x$ to $4x$) has the effect of moving the parabola to the right, which also increases the roots. Thus the parabola with the smallest coefficients has the largest positive root, hence $\boldsymbol{B}$ is correct.

3. Suppose we don't make any moves of size 25. Then the best we can do is to move five 11's to the right followed by three 18's to the left, which is eight moves. If we move one 25 to the right, it takes at least eleven more moves to reach the asterisk, but if we move one 25 to the left, followed by an 18 to the left and four 11's to the right, we land on the asterisk in only six moves. The same can be accomplished by moving two 25's to the left, then three 11's and one 18 to the right, but all other combinations of moves involve more steps, so **6** is the best possible.

4. Perhaps the most efficient and entertaining route to the answer is to utilize educated guesswork. Using the distance formula we determine that the sides of the regular tetrahedron have length $\sqrt{18}$. So to reach the fourth vertex we may begin at any of the given vertices and move ± 4, ± 1 and ± 1 units in the x, y and z-directions in any order, or move ± 3, ± 3 and 0 units instead. Some experimentation soon gives the answer.

One could also compute the normal vector $(1,-5,-1)$ to the plane defined by the three given points using the cross-product. (Details are left to the reader.) The Pythagorean Theorem shows that the height of a regular tetrahedron having side length $\sqrt{18}$ is $2\sqrt{3}$, while our normal vector above has length $3\sqrt{3}$. So we scale the normal vector by $\frac{2}{3}$ to get $(\frac{2}{3}, -\frac{10}{3}, -\frac{2}{3})$, then add or subtract it to the centroid $(\frac{7}{3}, \frac{4}{3}, \frac{8}{3})$ of the given points. This yields both possible fourth vertices; the one with integer coordinates is $\mathbf{(3,-2,2)}$.

5. In order to solve this problem we need to find the remainder that 2^n leaves when divided by 7. For $n = 0$ to 6 these remainders are 1, 2, 4, 1, 2, 4, 1, ..., repeating every three terms. (We leave it to the reader to learn the technique for finding such a pattern of remainders.) Hence $2^n - 2$ will be divisible by 7 exactly when n is 1 more than a multiple of 3. A similar analysis shows that $4^n - 4$ is divisible by 9 for the same values of n. Meanwhile, $6^n - 6$ is divisible by 11 precisely when n is 1 more than a multiple of 10, and $8^n - 8$ is divisible by 13 exactly when n is 1 more than a multiple of 4. Putting all these facts together, we find that n must be 1 more than a multiple of 60. The smallest such value of n is 1; the next smallest value is **61**.

6. The Cauchy-Schwarz inequality for four variables states that

$$(a_1^2 + a_2^2 + a_3^2 + a_4^2)(b_1^2 + b_2^2 + b_3^2 + b_4^2) \geq (a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4)^2,$$

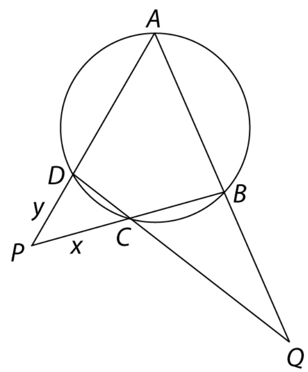
with equality if and only if the sequences (a_1, a_2, a_3, a_4) and (b_1, b_2, b_3, b_4) are proportional. Choosing $a_1 = w$, $b_1 = 1$, $a_2 = x/\sqrt{2}$, $b_2 = 2\sqrt{2}$, $a_3 = y/\sqrt{3}$, $b_3 = 3\sqrt{3}$, and $a_4 = z/2$, $b_4 = 8$ gives

$$\left(w^2 + \frac{x^2}{2} + \frac{y^2}{3} + \frac{z^2}{4}\right)(1 + 8 + 27 + 64) \geq (w + 2x + 3y + 4z)^2.$$

But we know that the right-hand side is $5^2 = 25$. Dividing through by 100 reveals that $w^2 + \frac{1}{2}x^2 + \frac{1}{3}y^2 + \frac{1}{4}z^2$ is at least $\frac{25}{100} = \mathbf{\frac{1}{4}}$.

7. A sketch of the situation is shown at right, where $AB = 4$, $BC = 2$, $CD = \sqrt{2}$ and $AD = 3\sqrt{2}$. We begin by noting that $\triangle APB \sim \triangle CPD$ since $ABCD$ is cyclic. Letting $PC = x$ and $PD = y$ we have

$$\frac{x}{y + 3\sqrt{2}} = \frac{y}{x+2} = \frac{\sqrt{2}}{4}.$$



Solving yields $PC = 2$ and $PD = \sqrt{2}$. In the same manner we find that $QB = 2$ and $QC = 2\sqrt{2}$. (Note that the diagram is not drawn to scale.) Next we realize that $\triangle PAB$ is a right isosceles triangle, since $PB = BA = 4$ and $PA = 4\sqrt{2}$. Hence $m\angle PAB = 45°$. Finally, applying the Law of Cosines to $\triangle PAQ$ gives

$$PQ^2 = (4\sqrt{2})^2 + 6^2 - 2(4\sqrt{2})(6)\cos 45° = 20,$$

so $PQ = \mathbf{2\sqrt{5}}$.

Observe that there was a shortcut to finding the answer—the alert solver may have noticed that $AB^2 + BC^2 = 20$ and $AD^2 + CD^2 = 20$ also. This implies that $\overline{AC}$ is a diameter of the circle and that $\angle ABC$ and $\angle ADC$ are both right angles.

December 2009

⋆ National Level ⋆

The Mandelbrot Competition

Round Two Solutions

Greater Testing Concepts
PO Box 760
Potsdam, NY 13676

The Mandelbrot Competition
www.mandelbrot.org
info@mandelbrot.org