February 2010

The Mandelbrot Team Play

Round Two Test

Time Limit: 60 minutes

Facts: An *algebraic identity* is an equality which asserts that two different expressions are identical, meaning that one expression can be transformed into the other using valid algebraic manipulations. In an algebraic identity the two sides will be equal for any values of the variables involved. By contrast, in an equation the two sides are equal only for certain values of the variables. For instance, $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$ is an algebraic identity, while $(x+y)^3 = x^3 + y^3$ is an equation, since in the latter case the two expressions are equal only when $x = 0$, $y = 0$ or $x = -y$.

Setup: In the questions below we will consider the following elegant algebraic identity. For real numbers a, b, c and d satisfying $ab = cd$, it is the case that

$$a^3 + b^3 + c^3 + d^3 = (a+b-c)^3 + (a+b-d)^3 + (c+d-a)^3 + (c+d-b)^3. \qquad (*)$$

Problems

Part i: (4 points) Let us choose $a = 1$, $b = 6$, $c = 2$, $d = 3$ so that $ab = cd$, as required. Confirm that both sides of $(*)$ are equal for these values. Now choose a new set of distinct values for a, b, c and d satisfying $ab = cd$ and show that $(*)$ holds in this case as well.

Part ii: (4 points) Find a way to write 1368 as a sum of four perfect cubes. Then use $(*)$ to help find a different way to write 1368 as a sum of four perfect cubes.

Part iii: (5 points) Suppose that $ab = cd$ as above. Find and verify an algebraic identity of the form $a^2 + b^2 + (c+d-a)^2 + (c+d-b)^2 = (\quad)^2 + (\quad)^2 + (\quad)^2 + (\quad)^2$.

Part iv: (5 points) By algebraically expanding the cubes and using the fact that $ab = cd$, prove that the expression on the right-hand side of $(*)$ reduces to $a^3 + b^3 + c^3 + d^3$.

Part v: (5 points) Explain how to find four fractions, none of which are integers, whose cubes sum to 42.

Part vi: (5 points) Prove that if a, b, c and d are distinct positive integers with $ab = cd$, then $a^3 + b^3 + c^3 + d^3$ factors into a product of at least four (not necessarily distinct) primes.