Team Play Solutions

Part i: We find by direct computation that

$$1^3 + 6^3 + 2^3 + 3^3 = 1 + 216 + 8 + 27 = 252.$$

Meanwhile, the quantities $(a+b-c)$, $(a+b-d)$, $(c+d-a)$, $(c+d-b)$ reduce to 5, 4, 4, and -1. So we next compute

$$5^3 + 4^3 + 4^3 + (-1)^3 = 125 + 64 + 64 - 1 = 252,$$

so the two sides agree. If we try $a = 1$, $b = 8$, $c = 2$, $d = 4$ we find that

$$1^3 + 8^3 + 2^3 + 4^3 = 7^3 + 5^3 + 5^3 + (-2)^3 = 585.$$

Similarly, when $a = 1$, $b = 10$, $c = 2$, $d = 5$ we have

$$1^3 + 10^3 + 2^3 + 5^3 = 9^3 + 6^3 + 6^3 + (-3)^3 = 1134.$$

Part ii: It makes sense to try 1000 as one of our perfect cubes, so it remains to write 368 as a sum of three cubes. Some experimentation yields $368 = 216 + 125 + 27$. Therefore we have

$$1368 = 10^3 + 6^3 + 5^3 + 3^3.$$

But this corresponds to using $a = 3$, $b = 10$, $c = 5$ and $d = 6$. Since $ab = cd$ our identity applies, yielding the second representation

$$1368 = 8^3 + 8^3 + 7^3 + 1^3.$$

Part iii: Based on the form of the identity involving cubes, we guess that the terms c^2 and d^2 ought to appear somewhere, as well as the terms $(a+b-c)^2$ and $(a+b-d)^2$. In fact these are exactly the correct terms to include on the right-hand side. Expanding the left side yields

$$2a^2 + 2b^2 + 2c^2 + 2d^2 + 4cd - 2ac - 2ad - 2bc - 2bd.$$

The expansion of the right side is virtually identical, except that the $4cd$ term is replaced by a $4ab$ term. However, since $ab = cd$ we conclude that the two sides are always equal.

Part iv: Expanding $(a+b-c)^3$ requires patience and care. The most obvious approach is to find $(a+b-c)^2 = a^2 + b^2 + c^2 + 2ab - 2ac - 2bc$ and then multiply this result by $(a+b-c)$ again. It is also possible to expand $(a+b-c)(a+b-c)(a+b-c)$ all at once. Regardless, the end result for $(a+b-c)^3$ looks like

$$a^3 + b^3 - c^3 + 3a^2b + 3ab^2 - 3a^2c + 3ac^2 - 3b^2c + 3bc^2 - 6abc.$$

Adding together similar expressions for $(a+b-d)^3$, $(c+d-a)^3$ and $(c+d-b)^3$ we discover that a number of terms cancel. For instance, the $3ac^2$ term will cancel with a $-3ac^2$ term in the expansion of $(c+d-a)^3$. This occurs for any cross term involving one of a or b and one of c or d. When the dust settles we are left with

$$\begin{aligned} &a^3 + b^3 + c^3 + d^3 \\ &\qquad + 6a^2b + 6ab^2 + 6c^2d + 6cd^2 - 6abc - 6abd - 6acd - 6bcd. \end{aligned}$$

Finally, the latter eight terms cancel in pairs due to the fact that $ab = cd$. For instance, we have $6a^2b - 6acd = 6a(ab - cd) = 0$. Hence the right-hand side of $(*)$ reduces to $a^3 + b^3 + c^3 + d^3$, as desired.

Part v: Our strategy is to find a way to write 42 as a sum of four integer cubes, then use $(*)$ to obtain a second representation involving cubes of fractions. It does not take long to discover that

$$42 = 3^3 + 2^3 + 2^3 + (-1)^3.$$

If we now let $a+b-c = 3$, $a+b-d = 2$, $c+d-a = 2$, $c+d-b = -1$ and solve for a, b, c and d we obtain $a = \frac{1}{3}$, $b = \frac{10}{3}$, $c = \frac{2}{3}$ and $d = \frac{5}{3}$. Fortunately $ab = cd$, so our identity applies, and we deduce the nice numerical result

$$\left(\frac{1}{3}\right)^3 + \left(\frac{2}{3}\right)^3 + \left(\frac{5}{3}\right)^3 + \left(\frac{10}{3}\right)^3 = 42.$$

Part vi: This result is motivated by the fact that all three of the numbers computed in the first part have lots of prime factors. For instance,

$$\begin{aligned} 252 &= 2 \cdot 2 \cdot 3 \cdot 3 \cdot 7, \\ 585 &= 3 \cdot 3 \cdot 5 \cdot 13, \\ 1134 &= 2 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 7. \end{aligned}$$

To explain why this happens we will algebraically factor the expression $a^3 + b^3 + c^3 + d^3$. Letting $N = ab = cd$, we begin by writing

$$a^3 + b^3 + c^3 + d^3 = (a^3 + c^3)\left(1 + \left(\frac{N}{ac}\right)^3\right).$$

While N^3 is certainly divisible by both a^3 and c^3 separately, it may not be divisible by their product if a and c have common factors. This difficulty can be remedied by letting $t = \mathrm{GCD}(a, c)$, then pulling out t^3 from the first factor and distributing it through the second factor to get

$$a^3 + b^3 + c^3 + d^3 = \left(\left(\frac{a}{t}\right)^3 + \left(\frac{c}{t}\right)^3\right)\left(t^3 + \left(\frac{Nt}{ac}\right)^3\right).$$

We leave it to the reader to confirm that Nt/ac is an integer.

To complete the argument we must show that each of these sums of distinct cubes has at least two prime divisors. But this follows because $m^3 + n^3 = (m + n)(m^2 - mn + n^2)$. The first factor is clearly 2 or greater, while the second is equal to $(m - n)^2 + mn$, which is also at least 2. Therefore each factor above has at least two prime divisors, yielding at least four prime divisors in all, as claimed.

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Round Two Solutions

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