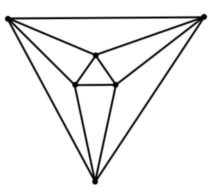
Answer Key	
1. 100	4. 18
2. 2314	5. 7/4
3. 8	6. 59
	7. 147

1. Working from the last set of parentheses back towards the first, we see that all the numbers in the 18th and 19th sets of parentheses cancel except for the number 19. The same occurs for the 16th and 17th terms, leaving just 17. When the dust settles, we have

$$1 + 3 + 5 + \cdots + 17 + 19 = 20 + 20 + 20 + 20 + 20 = \mathbf{100}.$$

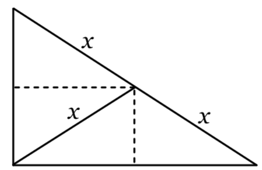
2. Suppose that $a > b$. This means that $a < c$, since exactly one of $a > b$ or $a > c$ is true. So far we have $b < a < c$, thus $b < c$. Condition (ii) now implies $b > d$, giving $d < b < a < c$ overall. But this violates condition (iii), so our original assumption that $a > b$ must be mistaken. Starting with $a < b$ instead and employing the same logic, we arrive at $c < a < b < d$ overall. Hence $c = 1$, $a = 2$, $b = 3$, $d = 4$ and our four-digit number $1000a + 100b + 10c + d$ must be **2314**.



3. Upon drawing a diagram we instinctively lean towards configurations with as few intersections among the segments as possible, since these create more regions. It is actually possible to obtain no intersections, as illustrated here, giving eight regions. In fact, Euler's formula ensures that every such diagram will involve precise eight regions. Each of our six points is the endpoint of four segments, for a total of 24 endpoints, hence there are 12 segments. According to Euler, $V - E + F = 2$ for such configurations, where F is the number "faces" (regions). In our case $V = 6$ and $E = 12$, so we must have $F = 8$. Therefore **8** regions is the best possible.

4. Some experimentation reveals that for any positive integer n one of $4 + n$, $6 + n$, $8 + n$ is composite. In retrospect, this makes sense because exactly one of them will be a multiple of 3. (Because $8 + n$ is a multiple of 3 exactly when $5 + n$ is.) Since 4, 6, 8 are the first three composite numbers, $f(n)$ will always be one of these three numbers. In fact, $f(2) = 4$, $f(3) = 6$ and $f(1) = 8$, so all three values occur, meaning that the desired sum is $4 + 6 + 8 = \mathbf{18}$.

5. It is possible to bash this algebraically, but there is a more elegant and illuminating geometric approach. Place the two isosceles triangles next to one another as shown. We claim that they must fit together to form a large right triangle. To see why, split the isosceles triangles along the dotted lines into two right triangles. Since all four small right triangles have the same area, we can rearrange the pieces of one isosceles triangle to form the other, which explains the claim above. By the Pythagorean theorem, we now deduce



$$(\sqrt{6})^2 + (\tfrac{5}{2})^2 = (2x)^2 \quad \Longrightarrow \quad 6 + \frac{25}{4} = \frac{49}{4} = 4x^2,$$

hence $x^2 = \frac{49}{16}$, which gives $x = \mathbf{\frac{7}{4}}$.

6. It is tempting to multiply out, but instead we employ the formula $1 + 2 + \cdots + n = \frac{1}{2}n(n + 1)$ to rewrite the expression as

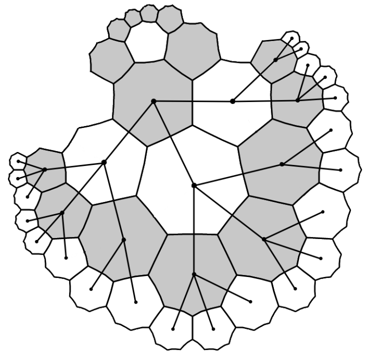
$$\tfrac{1}{2}(1)(2) \cdot \tfrac{1}{2}(2)(3) \cdot \tfrac{1}{2}(3)(4) \cdots \tfrac{1}{2}(59)(60).$$

Combining corresponding terms, we discover that we are dealing with two factorials divided by a large power of 2; namely

$$\frac{(59!)(60!)}{2^{59}} = \frac{2(59!)(60!)}{2^{60}},$$

where we have included the extra 2 to capitalize on the fact that $2^{60} \equiv 1$ mod 61 by Fermat's Little Theorem. Moreover, by Wilson's Theorem $60! \equiv -1$ and $59! \equiv 1$ mod 61, hence our entire expression reduces to just $-2 \equiv 59$ mod 61. Therefore the desired remainder is **59**.

7. Connect the center of each heptagon in ring 2 or higher to the center of the adjacent heptagon in the previous ring. If there are two options, choose the one furthest around counterclockwise. This forms seven trees, one of which is shown in the diagram. Note that there are two types of nodes: 2-branch nodes, which correspond to heptagons that touch two



tiles in the previous ring, and 3-branch nodes, which correspond to tiles with only one neighbor in the previous ring. Also note that the rightmost branch of any node leads to a 2-branch node, while the others lead to 3-branch nodes. This means that if a layer of a tree has a 3-branch and b 2-branch nodes, the next layer has $(2a+b)$ 3-branch and $(a+b)$ 2-branch nodes. By iterating this recurrence (or by inspecting the diagram), we find that the third and fourth layer of each tree has 8 and 21 tiles. Since there are 7 trees, the fourth ring has $7(21) = \mathbf{147}$ tiles.

N.B. The alert reader may have noticed that Fibonacci numbers make an appearance. This pattern persists, so the general formula for the number of heptagons in the nth ring is $7F_{2n}$.

Problem Credits

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February 2018

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Round Five Solutions

Produced by
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