Answer Key

1. 4
2. 4×10^{29}
3. 18
4. 3410
5. 36
6. $1963\frac{50}{101}$
7. 30π

1. There are exactly four such lines. A line through Q parallel to any of the other three lines will intersect the rest of the diagram in exactly two other points. And drawing the line from Q through the point common to all three given lines results in only one point of intersection with the rest of the diagram. Therefore the total is **4** lines.

2. We compute the ratio as follows:

$$\frac{2 \times 10^{50}}{5 \times 10^{20}} = \frac{2}{5} \times 10^{30} = 0.4 \times 10^{30} = \mathbf{4 \times 10^{29}}.$$

(We "borrow" a 10 from 10^{30} to write the answer in the desired form.)

3. Let t_1, t_2 and t_3 represent Sutton's test scores in order from least to greatest. Then the difference A between his higher two test scores is $A = t_3 - t_2$, and similarly $B = t_2 - t_1$. We wish to compute $A - B$, or $(t_3 - t_2) - (t_2 - t_1) = t_3 - 2t_2 + t_1$. We are told that his average $\frac{1}{3}(t_1 + t_2 + t_3)$ is 6 more than his middle score t_2. Thus

$$\frac{t_1 + t_2 + t_3}{3} = t_2 + 6 \qquad \Longrightarrow \qquad t_1 + t_2 + t_3 = 3t_2 + 18.$$

Subtracting $3t_2$ from both sides yields $t_3 - 2t_2 + t_1 = \mathbf{18}$.

4. Suppose the length and width of our rectangle are x and y. Then there are $(x-1)$ vertical and $(y-1)$ horizontal grid lines within the rectangle, for a total of $(x-1)(y-1)$ grid points inside the rectangle. Furthermore, each of the $(x-1)$ vertical grid lines is split into y unit segments, while each of the $(y-1)$ horizontal grid lines is split into x unit segments. Hence the total score of such a rectangle would be

$$7(x-1)(y-1) - 3y(x-1) - 3x(y-1) = xy - 4x - 4y + 7.$$

To obtain a total score of 2013 we must have

$$xy - 4x - 4y + 7 = 2013 \qquad \Longrightarrow \qquad (x-4)(y-4) = 2022.$$

But $2022 = 2 \cdot 3 \cdot 337$, and we must have $x, y \geq 8$, which forces us to use $x = 10$, $y = 341$ or vice-versa. Either way the area is $(10)(341) = \mathbf{3410}$.

5. Imagine making a list of the divisors of each number from 1 to 45, with each set of divisors on a separate line. Any number 23 or greater will only appear once, so such a number is a good candidate for the least likely result. Also note that the more divisors a number has, the less likely that any particular divisor of that number will be chosen. A bit of legwork reveals that the positive integer from 1 to 45 with the most divisors is 36, with nine divisors. Therefore **36** is the least likely result: it appears with a probability of only $\frac{1}{45} \cdot \frac{1}{9}$.

6. The key to an efficient solution is realizing that all the solutions to the equation $x + 100\langle x \rangle = 2013$ are evenly spaced. Therefore it suffices to find only the smallest and largest solutions, then average those. To see that the solutions are evenly spaced, observe that the value of $f(x)$ at $x = a$ is (usually) the same as its value at $x = a + \frac{100}{101}$, since

$$\begin{aligned}(a + \tfrac{100}{101}) + 100\langle a + \tfrac{100}{101} \rangle &= (a + \tfrac{100}{101}) + 100(\langle a \rangle - \tfrac{1}{101}) \\ &= a + 100\langle a \rangle.\end{aligned}$$

So if $f(x) = 2013$ for $x = a$, then $f(x) = 2013$ at $x = a + \frac{100}{101}$ also, then again at $x = a + \frac{200}{101}$, and so on. This reasoning eventually breaks down, once both a and $a + \frac{100}{101}$ both fall between the same pair of consecutive integers, at which point we have reached the last solution.

There is no solution below $x = 1913$, since $100\langle x \rangle$ is at most 100. Trying a number of the form $x = 1913 + \delta$ one finds that we need $\delta = \frac{100}{101}$, giving $x = 1913\frac{100}{101}$ as the smallest solution. Clearly the largest solution is $x = 2013$. Hence the average is

$$\tfrac{1}{2}\left(1913\tfrac{100}{101} + 2013\right) = \mathbf{1963\tfrac{50}{101}}.$$

7. To make any real headway with this problem one needs to be familiar with the concept of a parallel projection. This is a transformation of the plane that "stretches" the diagram in various directions. For now, we note that a parallel projection preserves ratios of areas, midpoints, tangencies, and turns circles into ellipses. We also mention the golden rule of parallel projections: given any three noncollinear points in a diagram there exists a parallel projection taking these three points to any other three noncollinear points.

So consider an equilateral triangle with side length 2, along with its inscribed circle, which is clearly tangent to the sides of the triangle at their midpoints. The triangle has area $\sqrt{3}$, while the inscribed circle has area $\pi/3$, as the reader may readily confirm. Now perform a parallel projection taking the vertices of this triangle to the vertices of a triangle having sides of length 15, 21 and 24. The circle will become an ellipse, but it will remain tangent to the midpoints of the triangle, according to the properties mentioned above. Furthermore, the percentage of the area of the triangle enclosed by the ellipse will also remain the same. We can calculate the total area via Heron's Formula,

$$K = \sqrt{(30)(15)(9)(6)} = 90\sqrt{3}.$$

Let E be the area of the ellipse. The constant ratio of areas implies that

$$\frac{\pi/3}{\sqrt{3}} = \frac{E}{90\sqrt{3}} \qquad \Longrightarrow \qquad E = \mathbf{30}\pi,$$

and you're done.

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⋆ National Level ⋆

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Round Five Solutions

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