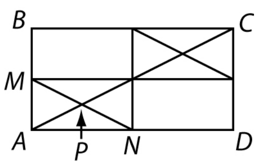
Answer Key

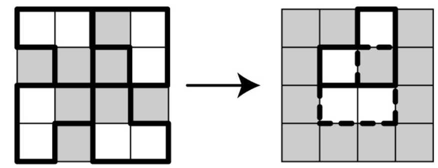
1.	$\frac{1}{3}$	4.	8
2.	9	5.	91
3.	6	6.	$\frac{5}{24}$
		7.	$\frac{11}{4}$

1. Segment $\overline{MN}$ and diagonal $\overline{AC}$ are drawn within rectangle $ABCD$, intersecting at P. Now imagine drawing the midpoints of the other two sides and connecting the various midpoints as shown. Clearly we will create four smaller rectangles, and the various segments will divide diagonal $\overline{AC}$ into four equal parts. Since $\overline{AP}$ is one of these parts while $\overline{PC}$ covers the remaining three, we will have $AP/PC = \mathbf{1/3}$.



2. Since we are not told which girl is twice as old as another we consider three cases. To begin, suppose that the ages of the girls are currently x, $2x$ and y. Notice that it will take longer for the oldest girl to become twice the age of the youngest as opposed to the middle girl. (Why?) So in one year their ages will be $x+1$, $2x+1$ and $y+1$ and we must have $y+1 = 2(2x+1)$. Four years later their ages will be $x+5$, $2x+5$ and $y+5$ and we will have $y+5 = 2(x+5)$. Solving these two equations for x and y yields $x=2$, $y=9$. If the ages of the girls were y, x and $2x$ instead, then the resulting equations give $x=4$, $y=1.5$, which doesn't work. Furthermore, the case x, y, $2x$ is impossible, according to the above note. Hence the oldest girl is currently **9**.

3. Clearly we will require at least four turns to flip over the four white corner squares. In the process we inevitably create some new white squares, but if we are careful it is possible to then finish the job with only two more flips. The diagrams at right show how this can be accomplished. (The final two L-shapes overlap; one of them is outlined with a dotted line to make its location more apparent.) Hence **6** flips suffice.



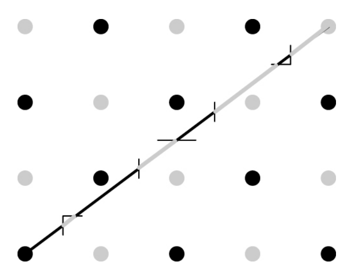
4. We first factor $84,000 = 2^5 \cdot 3 \cdot 5^3 \cdot 7$. There are many ways to write 84,000 as a product mn, but if m and n are to be relatively prime, then all five factors of 2 must present in either m or n, so that m and n don't have a common factor of 2. For the same reason all three factors of 5 must occur in either m or n. This leads to **8** distinct products:

$$(1)(2^5 \cdot 3 \cdot 5^3 \cdot 7),\ (2^5)(3 \cdot 5^3 \cdot 7),\ (3)(2^5 \cdot 5^3 \cdot 7),\ (5^3)(2^5 \cdot 3 \cdot 7),$$

$$(7)(2^5 \cdot 3 \cdot 5^3),\ (2^5 \cdot 3)(5^3 \cdot 7),\ (2^5 \cdot 5^3)(3 \cdot 7),\ \text{and}\ (2^5 \cdot 7)(3 \cdot 5^3).$$

5. If we begin computing values of $g(n)$ for $n = 1, 2, 3, \ldots$ it seems at first as though no output will be repeated. This is in fact the case for two-digit numbers: let $m = 10a + b$ and $n = 10c + d$ be two digit numbers; then $g(m) = (10a+b)+a+b = 11a+2b$ and $g(n) = 11c+2d$. If it were the case $g(m) = g(n)$ then we would have $11(a-c) = 2(d-b)$. But digits b and d can't differ by a multiple of 11 unless $b = d$, which means that $a = c$ also, which forces m and n to be the same number. However, as soon as we reach three digit numbers there is a repeated value, since $g(91) = 91+9+1 = 101$ and $g(100) = 100+1+0+0 = 101$ also. Hence **91** is the smallest value of n that gives a repeated output.

6. As we move away from the origin clearly the points on our segment will be blue initially. At some stage the points on the segment will be closer to $(1,0)$ than to $(0,0)$; this happens precisely when the segment crosses the vertical line $x = \frac{1}{2}$. The points quickly revert back to blue though; as soon as the segment crosses the horizontal line



$y = \frac{1}{2}$ which divides points closer to $(1, 0)$ from points closer to $(1, 1)$. The diagram above illustrates the various points at which the color of the segment changes; clearly the shortest such segment is the red one near the origin just described. (A symmetric blue segment having the same shortest length occurs near the points $(4, 3)$.) Since the equation of the line segment is $y = \frac{3}{4}x$ one easily finds that the endpoints of the short red segment are $(\frac{1}{2}, \frac{3}{8})$ and $(\frac{3}{2}, \frac{1}{2})$, so its length is **5/24**, using the distance formula.

7. Using the standard double-angle formulas $\cos(2\theta) = \cos^2\theta - \sin^2\theta$ and $\sin(2\theta) = 2\sin\theta\cos\theta$ we may rewrite the given expression as

$$\frac{41}{16} = 4\cos(2\theta) + 3\sin(2\theta) = 4\cos^2\theta + 6\sin\theta\cos\theta - 4\sin^2\theta.$$

This expression vaguely resembles $(3\cos\theta + \sin\theta)^2$; it has the correct general form and the correct cross-term. If we add $5\cos^2\theta + 5\sin^2\theta = 5$ to both sides then we obtain precisely that perfect square:

$$5 + \frac{41}{16} = 9\cos^2\theta + 6\sin\theta\cos\theta + \sin^2\theta = (3\cos\theta + \sin\theta)^2.$$

Taking square roots gives $3\cos\theta + \sin\theta = \sqrt{121/16} = \mathbf{11/4}$. (We choose the positive square root since θ is in the first quadrant.)

March 2010

⋆ Regional Level ⋆

The Mandelbrot Competition

Round Five Solutions

Greater Testing Concepts
PO Box 760
Potsdam, NY 13676

The Mandelbrot Competition
www.mandelbrot.org
info@mandelbrot.org