

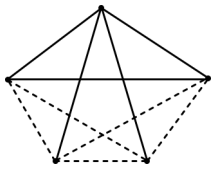
ANSWER KEY		4.	F_{703}
1.	$1\frac{2}{13}$	5.	176
2.	3	6.	6, 5, 9
3.	$3\sqrt{2}$	7.	$1620/\pi$

1. Converting each mixed fraction to an improper fraction, then flipping each fraction in the denominator, leads to

$$\left(\frac{7}{2} \cdot \frac{13}{3} \cdot \frac{21}{4} \cdots \frac{73}{8}\right) \left(\frac{4}{13} \cdot \frac{5}{21} \cdot \frac{6}{31} \cdots \frac{10}{91}\right) = \frac{7 \cdot 9 \cdot 10}{2 \cdot 3 \cdot 91},$$

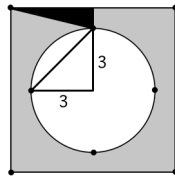
after a delightful amount of cancellation. This reduces to $\frac{15}{13}$, or $1\frac{2}{13}$, since it feels appropriate to write our answer as a mixed fraction.

2. There's exactly one way to use the five red (dotted) segments to form two red triangles, by having the triangles share one side. One such configuration is shown at left, but all others are equivalent to this one, by rearranging vertices. But now it's only possible to obtain one more blue (solid) triangle, bringing our total to **3** monochromatic triangles at most.



3. With a bit of trial and error we discover the unique arrangement of eight points within region \mathcal{R} for which each pair of points are no closer than $\sqrt{17}$ of one another, shown at right, using the four corners of the square and the four "compass points" of the circle.

Indeed, the shaded right triangle has legs of length 1 and 4, so the hypotenuse has length $\sqrt{17}$. By inspecting the diagram, we see that the next shortest distance will occur using two adjacent points on the circle. By the Pythagorean Theorem, this distance is $\sqrt{3^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$.



4. Of the many possible ways to simplify this expression, we choose to use the fact that $F_{701} + F_{700} = F_{702}$ to write

$$\begin{aligned} F_{705} - 3F_{701} - 2F_{700} &= F_{705} - F_{701} - 2(F_{701} + F_{700}) \\ &= F_{705} - F_{701} - 2F_{702}. \end{aligned}$$

But now $-F_{701} - F_{702}$ combines to give $-F_{703}$. In this manner the latter expression further reduces to

$$F_{705} - F_{703} - F_{702} = F_{705} - F_{704} = \mathbf{F_{703}}.$$

5. There are 7 choices for which letter to remove, then 26 possible letters that could replace it, so it would appear we have $7 \cdot 26 = 182$ sets of letters that could result. The catch is that seven of these sets are actually identical: we could remove F then replace it with F, or remove R and replace it with R, and so forth, each time getting back to 'FRIENDS' where we began. Since these seven sets only represent one distinct set of letters, our total should be reduced to $182 - 6 = \mathbf{176}$.

6. Notice that the bottom pair of equations cannot both be true, since if they were we would have

$$3x + 2z + 12 = 6y, \quad z + 6 = 3y \implies 3x = 0,$$

by subtracting twice the second equation from the first. This would give $x = 0$, when all values should be nonzero. For the same reason the left pair of equations cannot both be true, since

$$3x + 2z + 12 = 6y, \quad 2y = x + 4 \implies 2z = 0,$$

by adding three times the second equation to the first. We deduce that the bottom left equation must be false, for if it were true then two other equations would have to be false. It is now routine to solve the remaining three equations: one efficient method is to substitute $x = 2y - 4$ and $z = 3y - 6$ into the top right equation. This leads to $\mathbf{y = 5}$, from which we get $\mathbf{x = 6}$ and $\mathbf{z = 9}$. (Written in order we have **6, 5, 9**.)

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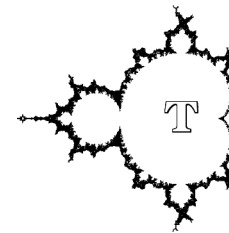
7. Illustrating this solution proved to be an insurmountable challenge, so the reader will have to use their imagination. Using standard facts relating the equation of an ellipse to its shape, we conclude that the graph of $\frac{x^2}{100} + \frac{y^2}{19} = 1$ will be an ellipse having a horizontal major axis with length $2\sqrt{100} = 20$ and vertical minor axis with length $2\sqrt{19}$. The foci are at $(\pm c, 0)$, where $c^2 = 100 - 19 = 81$, and hence $c = 9$. In other words, the foci are a distance of 18 apart, so when the paper is rolled up into a cylinder the circular cross-section will have circumference 18. But this means its radius r satisfies $2\pi r = 18$, so $r = 9/\pi$. Therefore its area equals $\pi r^2 = 81/\pi$, which results in a volume of $20 \cdot (81/\pi) = \mathbf{1620/\pi}$.

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The Mandelbrot Competition

Round Four Solutions