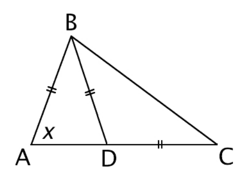
Answer Key	
1. 8	4. 15
2. -5	5. 10
3. $74°$	6. $14\frac{1}{3}$
	7. $\frac{2950}{1539}$

1. We may put A and B either in the group with C or in the group with D. In the former case, there is only one person out of E, F, G, and H remaining to include in the group with C, leading to four ways to split up the people. Placing A and B with D instead again leads to four possibilities, for a total of **8** ways to form the groups.

2. We can immediately deduce that $5 + \frac{4}{3+x}$ must equal 3, because 6 divided by this quantity gives 2. Therefore $\frac{4}{3+x} = -2$, which gives $3 + x = -2$. Finally, we discover that $x = -5$.

3. It will be helpful to label $m\angle BAC = x$, as shown in the diagram at left.



Since $\triangle BAD$ is isosceles, this means that $m\angle BDA = x$ also, so $m\angle BDC = 180 - x$. Therefore the remaining two angles of $\triangle BDC$ must add up to x, which means that each angle is $\frac{1}{2}x$, since $\triangle BDC$ is isosceles as well. In summary, the angles of $\triangle ABC$ measure $69°$, x, and $\frac{1}{2}x$, so

$$69° + x + \tfrac{1}{2}x = 180° \qquad \Longrightarrow \qquad \frac{3x}{2} = 111° \qquad \Longrightarrow \qquad x = \mathbf{74}°.$$

4. Let m be the number of marbles in the jar, and say that n of them are orange. If we draw a marble at random from the jar, the probability that it is orange is $\frac{n}{m}$. There are now $n-1$ orange marbles left among $m-1$ marbles, so the probability that the second draw is also orange is $\frac{n-1}{m-1}$. Thus we need

$$\frac{n}{m} \cdot \frac{n-1}{m-1} = \frac{1}{5} \qquad \Longrightarrow \qquad 5n(n-1) = m(m-1).$$

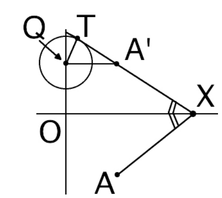
In other words, we need two products of consecutive integers, one of which is five times the other. Experimenting with various small values of n reveals that $5(6 \cdot 7) = 14 \cdot 15$, so we take $n = 7$ and $m = \mathbf{15}$. (There is a slightly involved procedure for finding *all* solutions to equations of this sort known as the theory of Pell equations.)

5. A quadratic equation has repeated roots exactly when its discriminant is zero. Writing the given equation as $x^2 + (z^5 + 2)x + (z^5 - 5)$ we see that it has the form $ax^2 + bx + c$ with $a = 1$, $b = z^5 + 2$, and $c = z^5 - 5$. The discriminant is

$$\begin{aligned} b^2 - 4ac &= (z^5 + 2)^2 - 4(z^5 - 5) \\ &= z^{10} + 4z^5 + 4 - 4z^5 + 20 \\ &= z^{10} + 24. \end{aligned}$$

Thus we need values of z for which $z^{10} = -24$. There are precisely **10** such values. (One can even list them all using roots of unity and $\sqrt[10]{24}$.)

6. The key to solving problems such as this is to take advantage of the equal angles by performing a reflection. In this case we reflect point $A(5, -7)$ across the x-axis to obtain $A'(5, 7)$, which must lie along the tangent line, as shown. Let Q be the center of the circle. Note that $A'Q = 5$, while $QT = 3$, hence $A'T = 4$ by the Pythagorean Theorem.



It now follows that the slope of line TA' is $-\frac{3}{4}$, using a pair of similar triangles. Since this line passes through $A'(5, 7)$, its equation is $y - 7 = -\frac{3}{4}(x - 5)$. We find the x-intercept by plugging in $y = 0$, yielding $\frac{3}{4}(x - 5) = 7$, or $x = \frac{28}{3} + 5 = \mathbf{14\frac{1}{3}}$.

7. Although not immediately evident, the graph of $f(x)$ is fractal-like in nature. We can capitalize on this fact to compute the desired area without actually summing complicated geometric sequences. At the heart of this approach is the observation that the graph for $0 \le x \le 1$ is precisely a replica of the graph for $0 \le x \le 10$, just scaled down by a factor of $\frac{1}{10}$. In other words, for $0 \le x \le 10$ we have $f(\frac{1}{10}x) = \frac{1}{10}f(x)$. This occurs

because the digits of $\frac{1}{10}x$ are essentially the same as those of x; only the decimal point moves. The upshot is that if we call the desired area A, then the area under the graph for $0 \le x \le 1$ is $\frac{1}{100}A$.

Note that the graph of $f(x)$ for $1 \le x \le 2$ is relatively tame: since $.88\overline{8} = \frac{8}{9}$ and $2.00\overline{0}$ are the nearest even-digit decimal numbers, the graph is piecewise-linear from $(\frac{8}{9}, 0)$ up to $(1\frac{4}{9}, \frac{5}{9})$, then back to $(2, 0)$. Hence the region for $1 \le x \le 2$ has area $\frac{49}{162}$. Combined with the above observation, we now know that the area for $0 \le x \le 2$ is $\frac{1}{100}A + \frac{49}{162}$.

But the graph of $f(x)$ for $2 \le x \le 4$ is an identical copy of the graph for $0 \le x \le 2$. In other words, $f(x+2) = f(x)$ in this range. Once again, this is a direct implication of the definition of $f(x)$ and the fact that 2 is an even digit. (The reader should supply the details.) The same holds true for $4 \le x \le 6$ and $6 \le x \le 8$. It almost works for $8 \le x \le 10$, except that 10 has an odd digit, so the graph from $(8\frac{8}{9}, 0)$ to $(10, 1\frac{1}{9})$ is a single line segment with slope one. Hence the area for $9 \le x \le 10$ is $\frac{99}{162}$ instead of $\frac{49}{162}$.

Putting everything together we find that

$$A = 5\left(\frac{A}{100}\right) + 4\left(\frac{49}{162}\right) + \left(\frac{99}{162}\right).$$

It is routine to solve this equation, giving the final answer of $\boldsymbol{\frac{2950}{1539}}$.

March 2012

⋆ National Level ⋆

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