Team Play Topics

Round Three

The first section of the Round Three Mandelbrot Team Play is reproduced below. A list of topics and practice problems are also provided to aid in preparation. Note that these problems are not meant to serve as a precise indicator of the problems that will appear on the contest. However, students who understand how to solve them should be able to make significantly more progress than they might have otherwise. So work hard on the problems, and good luck on the Team Play!

Facts: The main results outlined on this Team Play rely on a variety of tools from geometry and trigonometry. These include the facts that an inscribed angle is equal to half its intercepted arc, that a quadrilateral $PQRS$ is cyclic if and only if $\angle PQS \cong \angle PRS$, and the Pythagorean Theorem. Trigonometric facts include right triangle trig, the Law of Cosines, and elementary identities such as $\sin(x+90°) = \cos x$ or $\cos(x+90°) = -\sin x$.

Topics: Pythagorean Theorem, tangent circles, Law of Cosines, basic trig identities, inscribed angles, cyclic quadrilaterals

Practice Problems

Given a triangle ABC draw the incircle; that is, the circle inside $\triangle ABC$ tangent to all three sides. Call its center I and its radius r. Next draw a second circle inside $\triangle ABC$ tangent to sides $\overline{AB}$, $\overline{AC}$, and the incircle. Call its center K and its radius ρ, and let J be the point where the circles touch. Draw the line through J tangent to both circles, and label point M where this line intersects side $\overline{AB}$. Finally, let the incircle and the second circle be tangent to side $\overline{AB}$ at points T and U, respectively. Be sure to draw a large diagram illustrating this setup before starting on the questions below.

1. Demonstrate that $TU = 2\sqrt{r\rho}$.

2. Explain why M is the midpoint of $\overline{TU}$.

3. Prove that $\overline{IM}$ is parallel to $\overline{JU}$.

4. Find length IM in terms of r and ρ.

5. Finally, show that $NK = \frac{1}{2}\sqrt{(r+\rho)(r+4\rho)}$, where N is the midpoint of $\overline{IM}$.

Hints and answers on the next page. $\Longrightarrow$

Team Play Topics

HINTS AND ANSWERS

1. Draw the perpendicular from K to radius $\overline{IT}$. Next use the Pythagorean Theorem on the resulting right triangle having hypotenuse $r + \rho$ and one leg of length $r - \rho$. Then argue that TU is the same length as the other leg of this right triangle.

2. Note that $MT = MJ$ since triangles $\triangle MIT$ and $\triangle MIJ$ are congruent. In the same manner, $MJ = MU$. Therefore $MT = MU$, so M is the midpoint. In particular, $MT = \sqrt{r\rho}$.

3. Observe that $\angle AKU \cong \angle AIT$, since both are complementary to half of angle $\angle A$, using right triangles $\triangle KUA$ and $\triangle ITA$. Furthermore, $m\angle AIM = \frac{1}{2}m\angle AIT$ because of the congruent triangles from the previous part, while $m\angle AJU = \frac{1}{2}m\angle AKU$ using properties of inscribed angles. Therefore $\angle AIM \cong \angle AJU$, which implies that segments IM and JU are parallel.

4. In right triangle $\triangle ITM$ we know that $IT = r$ and $MT = \sqrt{r\rho}$. By the Pythagorean Theorem we now find that $IM = \sqrt{r(r + \rho)}$.

5. Apply the Law of Cosines to $\triangle NIK$, using $IN = \frac{1}{2}\sqrt{r(r + \rho)}$ and $IK = r + \rho$. Observe that $\cos(\angle KIN) = r/\sqrt{r(r + \rho)}$ since $\angle KIN \cong \angle MIT$. The result now follows after a bit of algebra. (There is a clever way to obtain the answer using a dilation followed by the Pythagorean Theorem. If you employed this approach, it will be worthwhile to rederive the result using the Law of Cosines, just for practice.)