Team Play Topics

Round Two

The first section of the Round Two Mandelbrot Team Play is reproduced below. A list of topics and practice problems are also provided to aid in preparation. Note that these problems are not meant to serve as a precise indicator of the problems that will appear on the contest. However, students who understand how to solve them should be able to make significantly more progress than they might have otherwise. So work hard on the problems, and good luck on the Team Play!

Facts: An *algebraic identity* is an equality which asserts that two different expressions are identical, meaning that one expression can be transformed into the other using valid algebraic manipulations. In an algebraic identity the two sides will be equal for any values of the variables involved. By contrast, in an equation the two sides are equal only for certain values of the variables. For instance, $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$ is an algebraic identity, while $(x+y)^3 = x^3 + y^3$ is an equation, since in the latter case the two expressions are equal only when $x = 0$, $y = 0$ or $x = -y$.

Topics: Algebraic identities, trinomial expansion, sums of cubes, factoring

Practice Problems

1. Algebraically expand the expression $(x+y)^3$.

2. Confirm that $(x+y)^3 = x^3 + y^3$ if and only if $x = 0$, $y = 0$ or $x = -y$.

3. Suppose that the variables x, y and z satisfy $x^2 + y^2 = z^2$. Establish the algebraic identity

$$(x+y+z)^2 + (x-y+z)^2 = 4z(x+z).$$

4. Find two different ways to write 3 as the sum of the cubes of three integers. (The integers are not necessarily distinct.)

5. Consider the identity $(a^2+b^2)(c^2+d^2) = (ac+bd)^2 + (ad-bc)^2$. Use this identity to predict how one may write $37 \cdot 41$ as a sum of two squares.

6. Confirm the identity in the previous part when $c = 1$, $d = 2$.

Hints and answers on the next page. $\Longrightarrow$

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HINTS AND ANSWERS

1. We have $(x+y)^2 = x^2+2xy+y^2$. Multiplying the latter expression by $(x+y)$ once more gives

$$(x+y)(x^2+2xy+y^2) = (x^3+2x^2y+y^2x)+(x^2y+2xy^2+y^3) = x^3+3x^2y+3xy^2+y^3.$$

2. The equation $(x+y)^3 = x^3+y^3$ reduces to just $3x^2y+3xy^2 = 0$ once we expand $(x+y)^3$ and subtract x^3 and y^3 from both sides. Factoring gives $3xy(x+y) = 0$, so we must have either $x = 0$, $y = 0$ or $x+y = 0$. The latter equality can be rewritten as $x = -y$, so we're done.

3. First one should expand the trinomial squares, yielding

$$(x^2+y^2+z^2+2xy+2xz+2yz)+(x^2+y^2+z^2-2xy+2xz-2yz).$$

Cancelling terms and combining the remaining terms gives $2x^2+2y^2+2z^2+4xz$. But $x^2+y^2 = z^2$, so this simplifies to just $2z^2+2z^2+4xz = 4z(x+z)$, as desired.

4. The obvious method is to write $3 = 1^3+1^3+1^3$. The slightly sneakier representation is given by $3 = 4^3+4^3+(-5)^3$.

5. Note that $37 = 1^2+6^2$ while $41 = 4^2+5^2$. Hence we may substitute $a = 1$, $b = 6$, $c = 4$ and $d = 5$ into the identity to obtain $37 \cdot 41 = 34^2+19^2$.

6. We wish to show that $(a^2+b^2)(5) = (a+2b)^2+(2a-b)^2$. Expanding the right-hand side leads to $(a^2+4ab+4b^2)+(4a^2-4ab+b^2) = 5a^2+5b^2$, which matches the left-hand side.