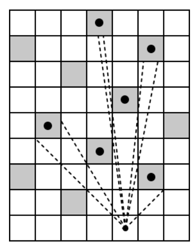
Answer Key	
1. 1007	4. 16
2. 24	5. 50/3
3. 7	6. 2/3
	7. 42

1. Instinctively, one wants to make the first and third number as small as possible in order to allow the middle number to become large. With this in mind, one soon hits upon the trio $1 + 1007 + 1009 = 2017$ or $2 + 1007 + 1008 = 2017$. We can't do better, since if the middle number were 1008 the smallest possible sum would be $1 + 1008 + 1009 = 2018$, which is too big. Hence **1007** is the largest possible middle number.

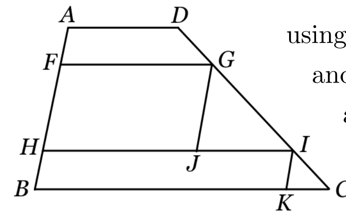
2. One convenient way to organize our work is to consider all shaded squares above the central dotted square, between lines with slope ± 1, as shown in the diagram. By carefully sweeping out this region from one side to the other, we see that exactly six shaded squares are visible from the central dot, as marked. By symmetry, the same thing occurs to the right, left, and below the central dot, for a total of **24** visible shaded squares.



3. Clearly it requires two turns to reach 4. From here it is tempting to multiply our way up to 64, then subtract 2 twice to obtain 60, in eight turns. (The sequence $0 \to 2 \to 4 \to 6 \to 12 \to 14 \to 28 \to 30 \to 60$ also does the job in eight turns.) But it's possible to improve on both these strategies by multiplying up to 32 in five turns, then subtracting 2, and finally multiplying by 2 to get to 60 in only **7** turns.

4. There are various algebraic methods for computing the perimeter of $ABCD$; here is a purely geometric approach. We first draw parallels to $\overline{AB}$ through F and H. Because of the parallelograms formed, we have $FG = HJ$ and $HI = BK$, for instance. Now observe that the difference between the perimeters of $ABCD$ and $AHID$ is

$$(AB + BC + CD + DA) - (AH + HI + ID + DA) = HB + KC + IC,$$



using the parallelograms. According to the givens and the similar triangles $\triangle IKC \sim \triangle GJI$, we also see that $HB = \frac{1}{2}(HF)$, $KC = \frac{1}{2}(JI)$, and $IC = \frac{1}{2}(GI)$. Hence the difference in perimeters is precisely $\frac{1}{2}(HF + JI + GI)$, which matches half the difference between the perimeters of $AFGD$ and $AHID$. In other words, perimeter$(ABCD) = 14 + \frac{1}{2}(14 - 10) = \mathbf{16}$.

5. Increasing a quantity by $p\%$ percent is tantamount to muliplying that quantity by $(1 + \frac{p}{100})$. For instance, an increase of 4% is accomplished by multiplying by 1.04. Therefore the statement of the problem is asking for the number q such that

$$\left(1 + \frac{2q}{100}\right)\left(1 + \frac{3q}{100}\right) = \left(1 + \frac{6q}{100}\right).$$

Multiplying out the lefthand side and canceling the 1s results in

$$\frac{2q}{100} + \frac{3q}{100} + \frac{6q^2}{10000} = \frac{6q}{100}.$$

We now multiply through by 100 and divide by q to get $2 + 3 + \frac{6q}{100} = 6$. Finally, rearranging leads to $\frac{6q}{100} = 1$, so $q = \boldsymbol{\frac{50}{3}}$ (or $16\frac{2}{3}$).

6. Finding the probability that *none* of the numbers $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$ lie between r and s turns out to be more accessible, so we tackle this question instead. For this to occur we must have both r and s between $\frac{1}{2}$ and 1, or both between $\frac{1}{4}$ and $\frac{1}{2}$, or both between $\frac{1}{8}$ and $\frac{1}{4}$, and so on. This total probability is given by

$$(\tfrac{1}{2})(\tfrac{1}{2}) + (\tfrac{1}{4})(\tfrac{1}{4}) + (\tfrac{1}{8})(\tfrac{1}{8}) + \cdots = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{1}{3},$$

since this a geometric series with first term $\frac{1}{4}$ and common ratio $\frac{1}{4}$ as well. Hence our desired probability is $1 - \frac{1}{3} = \boldsymbol{\frac{2}{3}}$.

7. To begin, let's determine how many complex numbers are solutions to both $z^{60} = 1$ and $z^{72} = 1$. The roots of these equations will be 60 (or 72) points evenly spaced around the unit circle, starting with 1. The first "interesting" point in common is $\frac{5}{60} = \frac{6}{72} = \frac{1}{12}$ of the way around, so there will be 12 points in common all together. In other words, the points in common will be precisely the $\text{GCD}(60, 72) = 12$ equally spaced points around the circle.

Extending this reasoning to the other pairs of equations, we have $\text{GCD}(60, 90) = 30$ common solutions for the first and third equations, and $\text{GCD}(72, 90) = 18$ common solutions to the second and third. Furthermore, the GCD of all three numbers is 6, which means we must subtract this from each pair's total to avoid counting solutions to all three equations. Our final answer is

$$(12 - 6) + (30 - 6) + (18 - 6) = \mathbf{42}.$$

Problem Credits

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