January 2011

The Mandelbrot Team Play

Round One Test

Time Limit: 60 minutes

Facts: Every positive integer can be written uniquely as a sum of distinct powers of 2. For example,

$$57 = 32 + 16 + 8 + 1 = 2^5 + 2^4 + 2^3 + 2^0.$$

We write this equality more compactly as $57 = 111001_2$, using 1's and 0's to indicate which powers of 2 to include or exclude from the sum, in order from largest to smallest. This is known as the *binary representation* of n. In a similar manner we find that $19 = 10011_2$, $31 = 11111_2$ and $32 = 100000_2$. It also makes sense to write $0 = 0_2$. Finally, recall that $\binom{r}{2}$ counts the number of ways to choose two objects from among a set of r objects.

Setup: For a positive integer n let $b(n)$ represent the number of 1's occurring in the binary representation of n. Equivalently, $b(n)$ is the sum of the binary digits of n. Thus we compute $b(57) = 4$, $b(19) = 3$, $b(31) = 5$, $b(32) = 1$ and $b(0) = 0$. In the questions below we will explore some unexpected results involving the values of $b(n)$.

Problems

Part i: (4 points) Construct a table listing each integer n from 0 to 31 in the first column. Then give the binary representation of n in the second column and the value of $b(n)$ in the third column. Use your table to compute $b(0) + b(1) + b(2) + \cdots + b(31)$.

Part ii: (4 points) Explain why $b(k) + b(31 - k) = 5$ for all $0 \le k \le 31$. Use this idea to find a formula for $b(0) + b(1) + b(2) + \cdots + b(2^r - 1)$ in terms of r.

Part iii: (5 points) Let m and n be positive integers satisfying $m + n < 2011$. Prove that we must have $b(m) + b(n) - b(m + n) < 11$. (Tip: perform the addition $m + n$ in binary.)

Part iv: (5 points) Let $v(k)$ count the number of factors of 2 in the positive integer k. Thus $v(12) = 2$ while $v(13) = 0$. Demonstrate that $v(1) + v(2) + \cdots + v(n) = n - b(n)$.

Part v: (5 points) Find an equality analogous to the previous part, but for base three. Clearly define any expressions used in your formula, then show that it works for $1 \le n \le 12$.

Part vi: (5 points) Demonstrate that for any $r \ge 2$ it is the case that

$$\binom{b(0)}{2} + \binom{b(1)}{2} + \binom{b(2)}{2} + \cdots + \binom{b(2^r - 1)}{2} = \binom{r}{2} 2^{r-2}.$$

Use this identity to find a formula for $b(0)^2 + b(1)^2 + b(2)^2 + \cdots + b(2^r - 1)^2$.