Answer Key	
1. 16	4. $2^{1008} \cdot 1009$
2. 70	5. 5
3. 18	6. 3/40000
	7. $\sqrt{3}$

1. It would be time-consuming to add up the positive differences for all sixteen possible paths. Instead, note that any path traverses all but one of the differences in the "outer ring" of squares, which sum to 18. That one unused difference is adjacent to the square we land on after the first step from the middle square. So now rather than a long eight-move path, we can minimize the expression

$$[\text{first positive difference}] - [\text{unused positive difference}] + 18$$

for a two-move path. This is much easier to check; the optimal path is to first travel from 1 to 4, then go clockwise to leave the 4–9 difference untraveled. This gives a minimum weight of $(4-1)-(9-4)+18 = \mathbf{16}$.

2. The painstaking means of solving this problem would be to break the trip into several segments and calculate the time that Rose waits for Karl at each trail marker. However, this work is made unnecessary by the observation that no matter where Rose waits, overall she waits the same amount of time. Rose walks the length of the trail in $\frac{7}{3}$ hours, not including the time in which she waits. Karl walks the length of the trail in $\frac{7}{2}$ hours. So Rose waits for Karl for $\frac{7}{2} - \frac{7}{3} = \frac{7}{6}$ hours, or **70** minutes.

3. From the first equation $\sin\alpha = \cos\beta$, we deduce that $\beta = 90° - \alpha$ because of the identity $\sin\alpha = \cos(90° - \alpha)$. From the second equation $\tan(2\alpha) = \tan(3\beta)$ we deduce that 2α and 3β differ by a multiple of $180°$, the period of the tangent function. Since the angles are acute, there are two possibilities: $2\alpha = 3\beta$ or $2\alpha = 3\beta - 180°$. For the first possibility $2\alpha = 3\beta$, we substitute $\beta = 90° - \alpha$ to get

$$2\alpha = 3(90° - \alpha) \;\Rightarrow\; \alpha = 54°.$$

However, that would mean $\beta = 90° - \alpha = 36° < \alpha$, which is undesirable. Using our second possibility $2\alpha = 3\beta - 180°$ instead yields

$$2\alpha = 3(90° - \alpha) - 180° \;\Rightarrow\; \alpha = \mathbf{18}°,$$

which is a valid solution as $\beta = 90° - \alpha = 72° > \alpha$.

4. The prime factorization of 2018 is $2 \cdot 1009$. Thus n, a multiple of 2018, must have 2 and 1009 in its prime factorization, say $n = 2^a 1009^b \cdots$. Applying the formula for number of divisors, we obtain $(a+1)(b+1)\cdots$. For this to equal $2018 = 2 \cdot 1009$, we must have $a, b = 1, 1008$ in some order, with no other factors. To minimize n, 1009 should have the smaller exponent, so our desired value of n is then $\mathbf{2^{1008} \cdot 1009}$.

5. One could examine all the cases of zero through ten people knowing the truth about cashews, but that approach doesn't bear much fruit without extensive calculations. Instead, consider any pair of people. The probability that they both know that cashews come from a fruit is $\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = \frac{1}{9}$. In other words, this pair contributes $\frac{1}{9}$ to the count, on average. Now, there are $\binom{10}{2} = 45$ pairs of people in the party, and since expected values add, the expected number of pairs in which both know that cashews come from a fruit is simply $(45)\left(\frac{1}{9}\right) = \mathbf{5}$.

6. There are too many fractions in this sum for a calculation by hand to be feasible. One work-around is to compute smaller cases; for instance, finding the sum from $\dfrac{1+2}{1^2 \cdot 2^2}$ to $\dfrac{(n-1)+n}{(n-1)^2 \cdot n^2}$ for small n. Calling the value of this sum $f(n)$, the original sum would then be $f(200) - f(100)$. Working out the first several values of $f(n)$ gives us the following table:

n	2	3	4	5	6
$f(n)$	$\frac{3}{4}$	$\frac{8}{9}$	$\frac{15}{16}$	$\frac{24}{25}$	$\frac{35}{36}$

Based on the pattern in the table we conjecture that $f(n) = 1 - \frac{1}{n^2}$. Using this, the original sum would be

$$f(200) - f(100) = \left(1 - \frac{1}{100^2}\right) - \left(1 - \frac{1}{200^2}\right) = \mathbf{\frac{3}{40000}}.$$

But why does this pattern appear? Each fraction in the sum is of the form $\frac{n+(n+1)}{n^2\cdot(n+1)^2} = \frac{2n+1}{n^2\cdot(n+1)^2}$. We can rewrite this as

$$\frac{2n+1}{n^2\cdot(n+1)^2} = \frac{(n^2+2n+1)-n^2}{n^2\cdot(n+1)^2} = \frac{(n+1)^2-n^2}{n^2\cdot(n+1)^2} = \frac{1}{n^2} - \frac{1}{(n+1)^2}.$$

Rewritten in this way, the entire sum becomes

$$\frac{1}{100^2} - \frac{1}{101^2} + \frac{1}{101^2} - \frac{1}{102^2} + \frac{1}{102^2} - \frac{1}{103^2} + \cdots + \frac{1}{199^2} - \frac{1}{200^2}.$$

Hence the sum "telescopes", with most adjacent terms canceling, leaving just the fractions on the ends.

7. We quickly obtain $m\angle BPC = 120^\circ$ and $m\angle APD = 60^\circ$ using the given information about the angles. Let $AP = a$, $BP = b$, $CP = c$ and $DP = d$. Using $\triangle APB$, $\triangle BPC$, $\triangle CPD$ and $\triangle APD$ and the Pythagorean Theorem or Law of Cosines, we get the four equations

$$a^2+b^2=36,\ b^2+c^2+bc=49,\ c^2+d^2=25,\ a^2+d^2-ad=16.$$

We are trying to find the positive difference between the areas of triangles $\triangle APD$ and $\triangle BPC$, which by the sine formula are $\frac{1}{2}ad\sin 60^\circ = \frac{\sqrt{3}}{4}ad$ and $\frac{1}{2}bc\sin 120^\circ = \frac{\sqrt{3}}{4}bc$, respectively. This difference, $\frac{\sqrt{3}}{4}|ad-bc|$, has terms which appear in the equations, and in fact we can attain this expression by taking the alternating sum of the equations in order, in which all squares magically cancel:

$$(a^2+b^2)-(b^2+c^2+bc)+(c^2+d^2)-(a^2+d^2-ad) =$$

$$= 36-49+25-16 \quad\Longrightarrow\quad ad-bc=-4.$$

Thus the difference between the areas is $\frac{\sqrt{3}}{4}|ad-bc| = \frac{\sqrt{3}}{4}|-4| = \boldsymbol{\sqrt{3}}$.

Problem Credits

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