February 2011

The Mandelbrot Team Play

Round Two Test

Time Limit: 60 minutes

Definitions: Let n be a positive integer. Given $2n$ equally spaced dots in a horizontal row, a *pairing* of the points consists of splitting the points into n pairs, then drawing n semicircles above the row joining each pair of dots. Two different pairings for $n = 3$ are pictured below. Semicircles are situated relative to one another in three different ways: they could *intersect* (if they cross over one another), they could be *nested* (if one is contained completely within the other), or else they could be *disjoint* (if their interiors do not overlap). Thus the pairing on the left involves nested and disjoint semicircles, but no intersection. The pairing on the right includes intersecting and disjoint semicircles, but no nesting.

Problems

Part i: (4 points) How many different pairings can be obtained for $n = 3$? Draw diagrams to illustrate all possible pairings, then label each with one or more of the letters I, N, and D to indicate whether that pairing involves Intersecting, Nested, or Disjoint semicircles. (Thus the pairings above should be labeled N, D and I, D from left to right.)

Part ii: (5 points) How many pairings for $n = 3$ do not involve any disjoint semicircles? Describe a method for obtaining all such pairings when $n = 4$ and explain why your method works. Then predict for any value of n how many pairings have no disjoint semicircles.

Part iii: (5 points) Label a row of ten dots as L, L, L, R, R, L, R, L, R, R from left to right. Consider pairings in which each point labeled 'L' is the left endpoint of a semicircle, and similarly for 'R' and right endpoints. Find, with proof, the number of such pairings.

Part iv: (4 points) Given a labeling of $2n$ dots with n L's and n R's, when can one create a pairing that "follows the labeling," as described in the previous part? State a condition on the positions of the L's and R's that indicates precisely when such a pairing is possible.

Part v: (5 points) Suppose we have a labeling of $2n$ dots for which there is at least one pairing that "follows the labeling," as described above. Prove that of all such pairings, there is exactly one that does not involve any intersecting semicircles.

Part vi: (5 points) Let n be a positive integer. For a row of $2n$ dots, prove that the number of pairings that do not involve any intersecting semicircles is equal to the number of pairings that do not involve any nested semicircles.