Answer Key

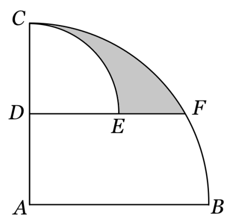
1. 15
2. $15\pi - 18\sqrt{3}$
3. $2\pi^2$
4. 21
5. 26
6. 90
7. 320

1. Some experimentation suggests that the optimal arrangement of digits is akin to the diagram at left, giving a minimal largest product of 15.

4	1	6
3	5	2

To see that it's not possible to go lower, note that the 6 and 5 cannot both be adjacent to only the 1 and 2. Hence at least one of them is next to a 3 or higher, meaning that some product will be at least 15 or higher; thus **15** must be the answer.

2. Since the radius of the circle is 12, we know $AF = 12$, and since D is a midpoint, we also have $AD = 6$. Hence $\triangle ADF$ is a $30°-60°-90°$ triangle, with $DF = 6\sqrt{3}$ and $area(ADF) = 18\sqrt{3}$. This means that $m\angle BAF = 30°$, so the area of sector BAF is a third of the entire quarter circle, giving an area of $\frac{1}{3}(\frac{1}{4}\pi 12^2) = 12\pi$. Finally, the area of quarter circle CDE is $\frac{1}{4}(\pi\, 6^2) = 9\pi$. Subtracting off all three of these areas from the original quarter circle CAB yields our answer of



$$\tfrac{1}{4}(\pi 12^2) - 9\pi - 12\pi - 18\sqrt{3} = \mathbf{15}\pi - \mathbf{18}\sqrt{\mathbf{3}}.$$

3. What is not apparent from the question is that the squares are tilted by 45° from the horizontal. However, this becomes clear upon realizing that taking $y = \frac{\pi}{2} + x$ gives $\cos y = \cos(\frac{\pi}{2} + x) = -\sin x$, so the line $y = \frac{\pi}{2} + x$ is part of our graph. The most efficient way to finish is to find and plot all the x- and y-intercepts. These occur where $\sin x + \cos 0 = 0$ and $\sin 0 + \cos y = 0$, so at the points $(-\frac{5\pi}{2}, 0)$, $(-\frac{\pi}{2}, 0)$, $(\frac{3\pi}{2}, 0)$ and $(0, -\frac{\pi}{2})$, $(0, \frac{\pi}{2})$, $(0, \frac{3\pi}{2})$, for instance. Sketching in the lines through these points with slope ± 1 reveals that each square has a diagonal of length 2π, and hence an area of $\frac{1}{2}(2\pi)^2 = \mathbf{2\pi^2}$.

4. Let the number of blocks of each color be a, b, c, d, where b represents the number of blue blocks. We are given that

$$ab + ac + ad + bc + bd + cd = 157, \qquad ac + ad + cd = 119.$$

Subtracting gives $b(a + c + d) = 38$. Since there is at least one block of each color, $a + c + d \geq 3$, meaning that either $b = 1$ or $b = 2$. But $b = 1$ leads to $a + c + d = 38$ and $ac + ad + cd = 119$, which has no integer solutions. (Try $a = 1$, 2 and 3 to see why.) We are left with $b = 2$ and $a + c + d = 19$, giving an overall sum of **21** blocks, for example with $a = 5$, $c = 7$, $d = 7$.

5. The chart below tabulates all permissible pairs a^b and c^d such that $a^b c^d = 3^6 5^8$. Its organization helps to make it clear that all possibilities are included. (In each column, let a^b be the exponential with the smaller base, to satisfy $a < c$.) There are a total of **26** ways in all.

$(3^1)^2$	$(3^1)^2$	$(3^1)^4$	$(3^1)^6$	$(3^1)^6$	$(3^1)^6$	$(3^2)^2$
$(3^2 5^4)^2$	$(3^1 5^2)^4$	$(3^1 5^4)^2$	$(5^1)^8$	$(5^2)^4$	$(5^4)^2$	$(3^1 5^4)^2$

$(3^2)^3$	$(3^2)^3$	$(3^2)^3$	$(3^3)^2$	$(3^3)^2$	$(3^3)^2$	$(3^1 5^1)^2$	$(3^1 5^1)^4$
$(5^1)^8$	$(5^2)^4$	$(5^4)^2$	$(5^1)^8$	$(5^2)^4$	$(5^4)^2$	$(3^2 5^3)^2$	$(3^1 5^2)^2$

$(3^1 5^1)^6$	$(3^2 5^1)^2$	$(3^2 5^1)^3$	$(3^3 5^1)^2$	$(3^3 5^1)^2$	$(3^3 5^1)^2$
$(5^1)^2$	$(3^1 5^3)^2$	$(5^1)^5$	$(5^1)^6$	$(5^2)^3$	$(5^3)^2$

$(3^2 5^2)^2$	$(3^2 5^2)^3$	$(3^3 5^2)^2$	$(3^3 5^2)^2$	$(3^3 5^3)^2$
$(3^1 5^2)^2$	$(5^1)^2$	$(5^2)^2$	$(5^1)^4$	$(5^1)^2$

6. In the language of congruences, we are looking for an integer D in the range $10 \leq D \leq 94$ such that

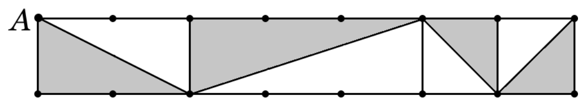
$$\frac{94!}{D} + D \equiv 0 \bmod 97 \quad \Longrightarrow \quad 94! \equiv -D^2 \bmod 97.$$

Multiplying through by (95)(96), which is the same as $(-2)(-1) \bmod 97$, yields

$$96! \equiv -2D^2 \bmod 97 \quad \Longrightarrow \quad -1 \equiv -2D^2 \bmod 97,$$

by Wilson's Theorem. Multiplying by -49 gives $D^2 \equiv 49 \bmod 97$, so $D = \pm 7$ are the solutions. We discard $D = 7$, since we need a two-digit number, leaving $D \equiv -7 \bmod 97$, or $D =$ **90** as our answer.

7. There is a clever, direct means to perform this count, which we will present. As suggested by the diagram, we may switch exactly once from shading triangles to the left of a diagonal segment to shading triangles to its right instead. This will occur at one of the eight possible positions



for a vertical divider. Our observation prompts the following strategy. First pick a vertical divider, then select any subset of the remaining seven spots (being sure to include the first and eighth) for the path to visit, then shade to the left or right accordingly. If the vertical divider is somewhere in the middle, this can be done in $6(2^5)$ ways; while if it is at an edge we have $2(2^6)$ ways, for a total of $192 + 128 =$ **320** ways.

Problem Credits

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