Answer Key

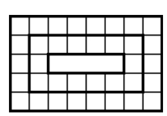
1. $1\frac{1}{2}$	4. 9696
2. 5	5. $3\sqrt{2}$
3. $\frac{1}{2}$	6. $\sqrt{5}-1$
	7. 9

1. Since three fire trucks can pump 1200 gallons in half a minute, each of them is able to pump 400 gallons in half a minute, thus five fire trucks can pump a total of 2000 gallons in half a minute. It is now clear that it will take three times as much time to pump 6000 gallons, which comes to $\mathbf{1\frac{1}{2}}$ minutes.

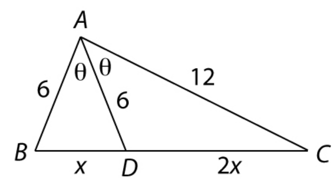
2. Let us first demonstrate that it is possible after five turns for all players to once again have ten marbles. This will occur if Al takes 1 marble from Cate and later 4 marbles from Betty. Meanwhile, Betty should take 2 marbles from Cate and later 5 marbles from Al. Finally, Cate should take 3 marbles from Betty. (The reader may verify that this solution works.) It is straight-forward to confirm that the task cannot be accomplished in one, two, three or four turns. Therefore it will take a minimum of **five** turns to return to their initial state.

3. There is no systematic way for solving an equation such as this in which the variable appears as an exponent and as a linear term, so we find solutions by inspection. (In other words, we guess.) It doesn't take long to find that $x = 1$ satisfies the equation. Clearly $x = 2$ is too large, while $x = 0$ is too small, so we next try $x = \frac{1}{2}$. This also works, since $9^{1/2} = 3$. Hence $x = \mathbf{\frac{1}{2}}$ is the smaller of the two answers.

4. The numbers of squares in the rectangular tracks form an arithmetic progression, so it is possible to compute the total by summing these terms using standard techniques. However, notice that each track fits neatly within the next larger one. Nesting them all within one another gives a solid 97×100 rectangle, with a small 1×4 rectangle missing in the middle. Therefore the total number of squares is equal to $(97)(100) - (1)(4) = \mathbf{9696}$.



5. The Angle Bisector Theorem states that $AC/AB = CD/BD$. We are given that $CD/BD = 2$ and that $AB = 6$, hence $AC = 12$. We also label $BD = x$ and $CD = 2x$, as shown in the diagram below. Since $\overline{AD}$ is the angle bisector, we may write $m\angle BAD = m\angle CAD = \theta$. The Law of Cosines now provides a means of solving for x. In $\triangle ABD$ we have



$$\cos\theta = \frac{6^2 + 6^2 - x^2}{2(6)(6)},$$

while in $\triangle ADC$ we have

$$\cos\theta = \frac{6^2 + 12^2 - (2x)^2}{2(6)(12)}.$$

Equating these expressions and multiplying through by 144 leads to

$$2(72 - x^2) = 180 - 4x^2 \quad \Longrightarrow \quad 2x^2 = 36 \quad \Longrightarrow \quad x = \mathbf{3\sqrt{2}}.$$

6. Translating the graph of $y = x^2$ by t units in the x-direction and $t-1$ units in the y-direction will shift the vertex from $(0,0)$ to $(t, t-1)$ so that it lies on the line $y = x - 1$. The new equation of the graph will be $y = (x-t)^2 + (t-1)$. Plugging in $y = 0$ gives the y-intercept of $t^2 + t - 1$. In order for this to also be an x-intercept, we should obtain $y = 0$ when substituting $x = t^2 + t - 1$. In other words, we must have

$$((t^2 + t - 1) - t)^2 + (t - 1) = 0 \quad \Longrightarrow \quad (t^2 - 1)^2 + (t - 1) = 0.$$

Factoring out $(t-1)$ leads to $t(t-1)(t^2+t-1) = 0$, so $t = 0$, 1, or $\frac{1}{2}(-1 \pm \sqrt{5})$. These give x-intercepts of -1, 1, 0 and 0, respectively. The average of the two x-intercepts is t, the x-coordinate of the vertex. Using this fact we quickly find the other x-intercept in each case to be 1, 1, $-1-\sqrt{5}$ and $\mathbf{-1+\sqrt{5}}$, the latter of which is the largest.

7. It will be convenient to work with congruences mod 41. To begin, notice that mod 41 we have $40 \equiv (-1)$, $39^2 \equiv (-2)^2$, $38^3 \equiv (-3)^3$, all the way up to $21^{20} \equiv (-20)^{20}$ mod 41. Combining like terms and all the negatives yields

$$\begin{aligned} 1^{40}2^{39}\cdots 39^2 40^1 &\equiv 1^{40}(-1)2^{39}(-2)^2\cdots 20^{21}(-20)^{20} \\ &\equiv 1^{41}2^{41}\cdots 20^{41}(-1)^{1+2+\cdots+20}. \end{aligned}$$

But $1+2+\cdots+20 = 210$ is even, so all the negatives cancel out. Furthermore, we know that $1^{41} \equiv 1$, $2^{41} \equiv 2$, …, $20^{41} \equiv 20$ by Fermat's Little Theorem. Therefore the remainder we are seeking is the same as $(1)(2)(3)\cdots(19)(20)$, reduced mod 41. This can be done by hand without much trouble by strategically combining certain pairs of terms. For instance, $(3)(14) \equiv 1$, $(5)(8) \equiv -1$, and so on. The reader is invited to finish reducing this product; if done correctly the result will be **9**.

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⋆ Regional Level ⋆

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