Team Play Solutions

Part i: We organize our list of partitions of $\{1,2,3,4\}$ according to the size of the subsets involved. For instance, there are four partitions of the form $\{a,b,c\}\{d\}$, so these are grouped together.

$\{1,2,3,4\}$	$\{1\}\{2\}\{3\}\{4\}$	$\{1,2,3\}\{4\}$
$\{1,2,4\}\{3\}$	$\{1,3,4\}\{2\}$	$\{2,3,4\}\{1\}$
$\{1,2\}\{3,4\}$	$\{1,3\}\{2,4\}$	$\{1,4\}\{2,3\}$
$\{1,2\}\{3\}\{4\}$	$\{1,3\}\{2\}\{4\}$	$\{1,4\}\{2\}\{3\}$
$\{2,3\}\{1\}\{4\}$	$\{2,4\}\{1\}\{3\}$	$\{3,4\}\{1\}\{2\}$

There are fifteen partitions in the list, so $B_4 = 15$.

Part ii: A careful inspection reveals that the seven partitions

$$\{1,2,3,4\} \quad \{1,3,4\}\{2\} \quad \{2,3,4\}\{1\} \quad \{1,2\}\{3,4\} \quad \{3,4\}\{1\}\{2\}$$

$$\{1\}\{2\}\{3\}\{4\} \quad \{1,2\}\{3\}\{4\}$$

remain unchanged when the 3 and 4 reverse places. In the first five cases the 3 and 4 appear in the same subset, while in the other two cases the 3 and 4 are singletons. The analogous partitions of $\{1,2,3,4,5\}$ are

$\{1,2,3,4,5\}$	$\{1\}\{2\}\{3\}\{4,5\}$	$\{1,2,3\}\{4,5\}$
$\{1,2,4,5\}\{3\}$	$\{1,3,4,5\}\{2\}$	$\{2,3,4,5\}\{1\}$
$\{1,2\}\{3,4,5\}$	$\{1,3\}\{2,4,5\}$	$\{1,4,5\}\{2,3\}$
$\{1,2\}\{3\}\{4,5\}$	$\{1,3\}\{2\}\{4,5\}$	$\{1,4,5\}\{2\}\{3\}$
$\{2,3\}\{1\}\{4,5\}$	$\{2,4,5\}\{1\}\{3\}$	$\{3,4,5\}\{1\}\{2\}$

$\{1,2,3\}\{4\}\{5\}$	$\{1,2\}\{3\}\{4\}\{5\}$	$\{1,3\}\{2\}\{4\}\{5\}$
$\{2,3\}\{1\}\{4\}\{5\}$	$\{1\}\{2\}\{3\}\{4\}\{5\}$	

These are obtained by taking each partition from part i. and replacing the '4' by '4,5' to ensure that the 4 and 5 are in the same subset. We also included all partitions of $\{1,2,3\}$ with the singletons $\{4\}\{5\}$ appended.

Part iii: If $\{n+1\}$ and $\{n+2\}$ must appear as singletons in our partition of $\{1,2,3,\ldots,n+2\}$ then the numbers from 1 to n will be grouped separately into various subsets. In other words, if we were to remove the singletons we would be left with all the partitions of $\{1,2,3,\ldots,n\}$. By definition there are B_n of these; appending $\{n+1\}$ and $\{n+2\}$ yields all the desired partitions of $\{1,2,3,\ldots,n+2\}$.

Now imagine that we have listed all partitions of $\{1,2,3,\ldots,n+2\}$ in which $n+1$ and $n+2$ appear in the same subset. If we replace each occurrence of '$n+1$, $n+2$' with just '$n+1$' then we will obtain all the partitions of $\{1,2,3,\ldots,n+1\}$. By definition there are B_{n+1} of these. Expanding each instance of '$n+1$' into '$n+1$, $n+2$' yields all the desired partitions of $\{1,2,3,\ldots,n+2\}$.

Part iv: To see this pairing in action we have listed the partitions of $\{1,2,3,4\}$ that correspond to one another upon swapping the 3 and 4.

$\{1,2,3\}\{4\}$	$\{1,2,4\}\{3\}$
$\{1,3\}\{2,4\}$	$\{1,4\}\{2,3\}$
$\{1,3\}\{2\}\{4\}$	$\{1,4\}\{2\}\{3\}$
$\{2,3\}\{1\}\{4\}$	$\{2,4\}\{1\}\{3\}$

The remaining seven partitions of $\{1,2,3,4\}$ are unaffected by the swap; these are listed in part ii.

Based on the previous parts, it should now be clear that the only partitions of $\{1,2,3,\ldots,n+2\}$ that are not affected by swapping $n+1$ and $n+2$ are those in which $n+1$ and $n+2$ are either singletons or appear in the same subset. Hence there are $B_{n+2} - B_{n+1} - B_n$ partitions remaining, all of which are paired up, so we deduce that this quantity is even. We already know that $B_1 = 1$ is odd, $B_2 = 2$ is even, while $B_3 = 5$ and $B_4 = 15$ are both odd. We can now predict that

$$B_5 = \text{even},\ B_6 = \text{odd},\ B_7 = \text{odd},\ B_8 = \text{even},\ B_9 = \text{odd},\ B_{10} = \text{odd},\ \ldots$$

For instance, we have seen that $B_5 - B_4 - B_3$ is even, and we know that B_4 and B_3 are odd, which forces B_5 to also be even. This pattern persists, so we can predict with certainty that B_n is even exactly when n is one less than a multiple of 3.

Part v: We claim that a partition of $\{1, 2, 3, \ldots, n+3\}$ is unaffected by the rotation $(n+1) \mapsto (n+2) \mapsto (n+3) \mapsto (n+1)$ in precisely two cases: those in which $\{n+1\}\{n+2\}\{n+3\}$ are singletons and those where $\{\ldots, n+1, n+2, n+3\}$ all appear in the same subset. For suppose that $n+1$ and $n+2$ are situated in different subsets. Then no other numbers can appear with them, otherwise we obtain a distinct partition when we replace $n+2$ by $n+1$. Similarly, $n+3$ must also be a singleton. On the other hand, suppose that $n+1$ and $n+2$ are situated in the same subset. Then $n+3$ must also appear in the same subset, otherwise we obtain a distinct partition when we replace $n+3$ by $n+2$.

In the first case we are left with a partition of $\{1, 2, 3, \ldots, n\}$ once we remove the singletons $\{n+1\}\{n+2\}\{n+3\}$, so there are B_n such partitions of this type. And in the second case we are left with a partition of $\{1, 2, 3, \ldots, n+1\}$ once we "collapse" $\{\ldots, n+1, n+2, n+3\}$ into just $\{\ldots, n+1\}$, so there are B_{n+1} such partitions of this type. (See the solution to part iii. for more details.) Hence there are a total of $B_n + B_{n+1}$ partitions of $\{1, 2, 3, \ldots, n+3\}$ unaffected by the rotation.

Part vi: The rotation $(n+1) \mapsto (n+2) \mapsto (n+3) \mapsto (n+1)$ has the effect of grouping all partitions of $\{1, 2, 3, \ldots, n+3\}$ into trios, except for those partitions unaffected by the rotation. For instance, the partitions below turn into one another when we rotate $4 \mapsto 5 \mapsto 6 \mapsto 4$.

$$\{1,3,6\}\{2,5\}\{4\} \quad \{1,3,4\}\{2,6\}\{5\} \quad \{1,3,5\}\{2,4\}\{6\}$$

There are $B_{n+3} - B_{n+1} - B_n$ partitions which are grouped into trios, hence this quantity is divisible by 3. In the language of congruences, we have $B_{n+3} - B_{n+1} - B_n \equiv 0 \bmod 3$, or $B_{n+3} \equiv B_{n+1} + B_n \bmod 3$. Using this recurrence and reducing mod 3 as we go, the sequence of Bell numbers looks like

$$1,\ 2,\ 2,\ 0,\ 1,\ 2,\ 1,\ 0,\ 0,\ 1,\ 0,\ 1,\ 1,\ 1,\ 2,\ 2,\ \ldots$$

At this point the sequence begins to repeat, so we conclude that B_n is a multiple of 3 exactly when $n \equiv 4, 8, 9, 11 \bmod 13$.

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