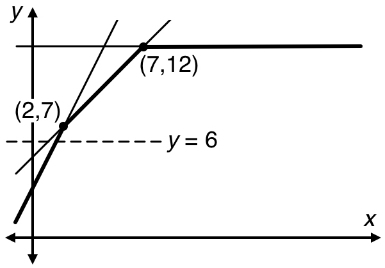
Team Play Solutions

Part i: By definition we know that $3 \odot x \odot x = 3 + x + x = 2x + 3$, and also $5 \odot x = x + 5$. Hence the given expression is equivalent to $\min(2x + 3, x + 5, 12)$. Therefore we graph the three lines $y = 2x + 3$, $y = x + 5$ and $y = 12$, then take the lower boundary of the region defined by these lines. The nonlinear points are located where $y = 2x + 3$ and $y = x + 5$ intersect, giving $(2, 7)$, and where $y = x + 5$ and $y = 12$ meet, giving $(7, 12)$. By including the dotted line $y = 6$ we see that it crosses the $y = 2x + 3$ component of the graph at $(1.5, 6)$, hence the given expression equals 6 when $x = 1.5$.



Part ii: If we assume that $\odot$ distributes over $\oplus$ in the same way that ordinary multiplication distributes over addition (which in fact it does, although we leave the routine verification of this fact to the reader), then it would stand to reason that

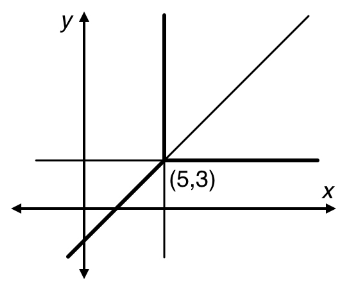
$$\begin{aligned}(x \oplus 4) \odot (x \oplus 7) &= (x \odot x) \oplus (x \odot 7) \oplus (x \odot 4) \oplus (4 \odot 7) \\ &= (x \odot x) \oplus ((4 \oplus 7) \odot x) \oplus (4 \odot 7).\end{aligned}$$

Since $4 \oplus 7 = 4$ and $4 \odot 7 = 11$, this reduces to just $x \odot x \oplus 4 \odot x \oplus 11$, as claimed. These quantities are in fact equivalent, and their graphs are identical. Similarly, applying this process to $(x \oplus 20) \odot (x \oplus 14)$ yields $x \odot x \oplus 14 \odot x \oplus 34$.

Part iii: Rewriting this tropical polynomial using the min function, we wish to find nonlinear points on the graph of $\min(3x, 2x + 3, x + c, 27)$. When $c = 8$, the four lines $y = 3x$, $y = 2x + 3$, $y = x + 8$ and $y = 27$ form a lower boundary that has bends at three distinct points: $(3, 9)$, $(5, 13)$ and $(19, 27)$. Hence the roots are $x = 3$, 5 and 19. Now observe that varying the value of c will raise (or lower) the line $y = x + c$, which may cause one of the nonlinear points to disappear. For instance, the lines $y = 2x + 3$ and $y = 27$ intersect at $(12, 27)$, so as soon as $c \geq 15$ the line $x + c$ will pass above this point and hence will no longer be part of the lower boundary, resulting in one less nonlinear point. In the same way, when $c \leq 6$ the line $y = x + c$ will pass below the existing nonlinear point $(3, 6)$, so $y = 2x + 3$ will no longer be part of the boundary, once again causing a decrease in the number of roots. In summary, for $c \geq 15$ or $c \leq 6$ there are only two distinct roots.

Part iv: The expression is symmetric, so we may assume without loss of generality that $a \leq b$. Note $(x \oplus a) \odot (x \oplus b)$ evaluates differently depending upon how x compares to a and b. If $x \leq a$, then $x \leq b$ also, so $x \oplus a = \min(x, a) = x$ and $x \oplus b = \min(x, b) = x$, so we obtain $x + x = 2x$ in this case. The cases $a \leq x \leq b$ and $x \geq b$ lead to values of $x + a$ and $a + b$, respectively. Hence the graph looks like $y = 2x$, then $y = x + a$, followed by $y = a + b$, with the changes occuring at $x = a$ and $x = b$, meaning that these are the roots.

Part v: We wish to analyze the graph of $z = \min(x + 5, y + 7, 10)$. This surface consists of the lower boundary of the solid region defined by the three planes $z = x + 5$, $z = y + 7$ and $z = 10$. The nonlinear points along this surface occur at the "creases" where two planes meet, so we next identify these locations. The first two planes intersect where $x + 5 = y + 7$, or above the line $y = x - 2$. Meanwhile, the other folds in the surface are situated where $x + 5 = 10$ and $y + 7 = 10$, or above the lines $x = 5$ and $y = 3$. These three lines are shown at left. The "triple point" $(5, 3)$ corresponds to the point $(5, 3, 10)$ in space where all three planes meet; the three creases in the surface extend outward as three rays from this central meeting spot. For instance, we only need the portion of the line $y = x - 2$ where $x \leq 5$ and $y \leq 3$ (highlighted with boldface in the diagram), since beyond this the values of $x + 5$ and $y + 7$ are above 10, and hence are no longer part of the lower boundary. This reasoning leads to the three boldface rays pictured, which are precisely the zero set.



Part vi: Although slightly counterintuitive, it is definitely *not* the case that the expressions $a \odot x \oplus b \odot y$ and $a \odot x \oplus b \odot y \oplus 0$ are equivalent, because the graph of the latter involves three planes meeting at a triple point, while the former involves only two planes. Hence the zero set of the former is the entire line where $x + a = y + b$, while the zero set of the latter consists of three rays emanating from a triple point. Performing the same analysis as in the previous part, we discover that the zero set of $a \odot x \oplus b \odot y \oplus 0$ consists of three rays extending out from the triple point $(-a, -b)$; one due north, one due east, and one exactly southwest. Therefore this is the general geometric form of a tropical line.

Now let P and Q be two arbitrary points in the plane. To begin, suppose the slope of line PQ is negative. If we place the triple point T so that the southwest ray from T passes through either of P or Q, then clearly it is impossible for either the north or east rays to pass through the other. So it must be the north and east rays which hit P and Q, which uniquely determines the position of T, directly below one of P and Q and to the left of the other.

In the same manner, if the slope of line PQ is between 0 and 1 then we must use a triple point to the left of one and northeast of the other, while if the slope of PQ is greater than 1 then the triple point must be below one point and northeast of the other. In each case the triple point (and hence the entire tropical line) is uniquely determined. The exceptions occur when line PQ is horizontal, vertical, or has slope 1. In these scenarios there are infinitely many positions for the tropical point, since a single ray can pass through both P and Q simultaneously.

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Round One Solutions

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