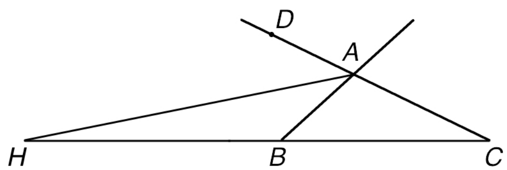
Answer Key	
	4. 10
1. 28	5. 5/12
2. 4°	6. \$2 1/7
3. 42	7. 93

1. Let Rohan and Arjun start with b beads. After the trade Rohan has seven more beads than before, so Arjun has seven less. If Rohan is to have twice as many beads we must have $b + 7 = 2(b - 7)$. Solving this equation gives $b = 21$, meaning that Rohan now has **28** beads.

2. Let the angle bisector of angle $\angle DAB$ meet line BC at H.



The Exterior Angle Theorem tells us that

$$m\angle DAB = m\angle ABC + m\angle ACB = 20^\circ + 12^\circ = 32^\circ.$$

The angle bisector splits this amount in half, so $m\angle HAB = 16^\circ$. Applying the Exterior Angle Theorem once again to triangle HAB, we find

$$m\angle AHB = m\angle ABC - m\angle HAB = 20^\circ - 16^\circ = \mathbf{4}^\circ.$$

3. The top left vertex can be any of three colors, while its neighbor to the right can then be any of the two remaining colors. So there are six possible ways to start the coloring; say we color them red and green, from left to right. Next, the middle two vertices (from top to bottom) cannot both be brown, so either one is brown or neither are.

Suppose the right middle vertex is brown. Then the left middle and bottom right vertices must both be green, leaving the bottom left to be either red or brown, giving two colorings. By symmetry there will also be two colorings when the left middle vertex is brown, bringing the total to four. Now suppose that neither of the middle vertices are brown; then the left middle must be green while the right middle is red. This leaves the bottom left to be red or brown, while the bottom right is green or brown. Any combination will work, except when they are both brown, giving three more colorings. Therefore the final total is $6(4 + 3) = \mathbf{42}$.

4. Consider the sides of the triangles and squares between the inner and outer borders. Observe that each triangle causes the next such side to be rotated $\frac{\pi}{13}$ radians relative to the previous, while each square results in no rotation, since the sides are parallel. Therefore in order for the sides to wrap completely around the loop we will need a total rotation of 2π radians, which will require 26 triangles. This leaves **10** squares.

5. We first compute that there are

$$\binom{10}{3} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$$

ways to choose three index cards. It is relatively straight-forward to list the cases in which the middle number is exactly between the others:

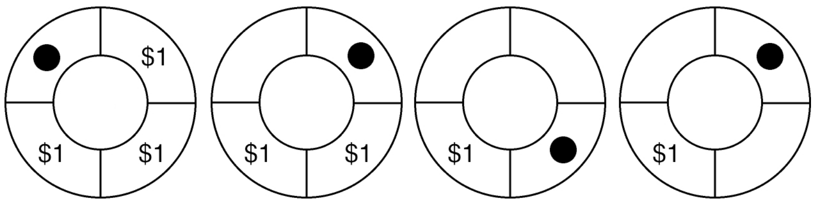
$$\{1, 2, 3\}, \quad \{1, 3, 5\}, \quad \{1, 4, 7\}, \quad \{1, 5, 9\}, \quad \{2, 3, 4\}, \ldots$$

We count twenty such cases in all, leaving 100 triples in which the middle number is actually closer to one of the others. By symmetry the middle number will be closer to the larger number in half of these cases, giving an answer of $\frac{50}{120} = \mathbf{\frac{5}{12}}$.

6. After the first move there are only four positions, up to rotation. (Other than game over.) These are shown below; let the expected total earnings for a game reaching these positions be E_1, E_2, E_3 and E_4 from left to right. Since every game immediately reaches the left-hand position, we are interested in the value of E_1. These expected values are related via linear equations. For instance, from position E_1 we reach position E_2 with probability $\frac{2}{3}$, while the game ends with only \$1 collected

with probability $\frac{1}{3}$. This implies that $E_1 = \frac{2}{3}(E_2) + \frac{1}{3}(1)$. Using similar reasoning we deduce that

$$\begin{aligned} E_1 &= \tfrac{2}{3}(E_2) + \tfrac{1}{3}(1) \\ E_2 &= \tfrac{1}{3}(E_2) + \tfrac{1}{3}(E_3) + \tfrac{1}{3}(2) \\ E_3 &= \tfrac{1}{3}(E_4) + \tfrac{1}{3}(3) + \tfrac{1}{3}(4) \\ E_4 &= \tfrac{2}{3}(E_3) + \tfrac{1}{3}(3). \end{aligned}$$



It is routine to solve these equations, giving $E_1 = \mathbf{2\frac{1}{7}}$.

7. Since both 1776 and 2012 are even the roots α and β will have the form $A \pm \sqrt{D}$ for certain integers A and D. (Their exact value is not important, only the fact that they are integers.) Furthermore, the sum of the roots of a quadratic of the form $x^2 + bx + c$ is $-b$, so we know that $\alpha + \beta = -2012$ and $\alpha^{101} + \beta^{101} = -B$. We next compute

$$(\alpha+\beta)^{101} = \alpha^{101} + 101\alpha^{100}\beta + \binom{101}{2}\alpha^{99}\beta^2 + \cdots + \beta^{101}.$$

We now make two important observations. First, if we were to substitute $\alpha = A+\sqrt{D}$ and $\beta = A-\sqrt{D}$ for all terms except the first and last, then every term involving $\sqrt{D}$ would cancel out. Secondly, since 101 is prime, all the remaining terms (which are integers) would reduce to 0 mod 101, using a well-known fact about Pascal's triangle mod 101. Therefore $(\alpha+\beta)^{101} \equiv \alpha^{101} + \beta^{101}$ mod 101. Using previous equalities, we have

$$(-2012)^{101} \equiv -B \bmod 101 \quad \Longrightarrow \quad B \equiv 2012^{101} \equiv 2012 \bmod 101,$$

by Fermat's Little Theorem. Since $2012 \equiv 93 \bmod 101$, we conclude that B leaves a remainder of **93**.

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⋆ National Level ⋆

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