Team Play Solutions

Part i: There are many lines with slope 1 that thread the lattice; for example, any line of the form $y = x + b$ having a y-intercept in the range $1 \leq b \leq 3$ will do, such as $y = x + \sqrt{2}$. (We omit the sketch of the line and the nearby lattice squares.) Note that there are plenty of such lines that cleanly thread the lattice.

Next we consider slope $-\frac{1}{2}$ lines. In this case any line of the form $y = -\frac{1}{2}x + b$ with $4 \leq b \leq 5$ works; all except the $b = 4$ and $b = 5$ intercepts cleanly thread. (There are other answers as well.) For slope $\frac{2}{3}$, however, we must choose the y-intercept carefully. Two possible answers are $y = \frac{2}{3}x + 1$ and $y = \frac{2}{3}x + 4\frac{1}{3}$. Every possible answer only barely threads the lattice, though. Finally, any line of the form $y = 3x + b$ with $1 \leq b \leq 2$ will thread the lattice; all except $b = 1$ and $b = 2$ cleanly thread. (Again, other answers are possible.)

Part ii: Suppose that we have drawn a line in the plane with slope m that cleanly threads the lattice. Consider a reflection of the entire plane over the horizontal line $y = 3$. It straight-forward to verify that the lattice squares wind up in exactly the same positions as before, while the reflected line has slope $-m$. Clearly the new line still cleanly threads the lattice, so we're done. The same type of argument can be used to show that there is also a line with slope $\frac{1}{m}$ that cleanly threads the lattice by reflecting the diagram over the line $y = x$. Finally, if we perform both of these reflections (in either order), we obtain a line with slope $-\frac{1}{m}$ that cleanly threads the lattice.

Part iii: The lower endpoint of the shadow is found by determining the equation of the line through the lower right corner of the $(5, 5)$-square, which has coordinates $(6, 5)$. This equation is $y = \frac{3}{4}x + \frac{1}{2}$. Since the y-intercept is $\frac{1}{2}$, the shadow cast on the y-axis must begin at $y = \frac{1}{2}$. In the same way we find that the line throught the upper left corner $(5, 6)$ is $y = \frac{3}{4}x + 2\frac{1}{4}$, so the upper endpoint of the shadow on the y-axis is $2\frac{1}{4}$. Hence, the length of the shadow is $2\frac{1}{4} - \frac{1}{2} = 1\frac{3}{4}$.

Part iv: Since the $(5, 5)$-square's shadow overlaps with that of the $(0, 0)$-square, it stands to reason that the $(10, 10)$-squares's shadow will overlap with that of the $(5, 5)$-square. Indeed, the lines with slope $\frac{3}{4}$ that just touch the $(10, 10)$-square on either side are $y = \frac{3}{4}x + 1\frac{3}{4}$ and $y = \frac{3}{4}x + 4$. Thus its shadow on the y-axis is the segment from $y = 1\frac{3}{4}$ to $y = 4$, which is higher than the shadow of the $(5, 5)$-square but overlaps it.

Consider the shadows of the squares along the main diagonal, which include the $(-5, -5)$, $(0, 0)$, $(5, 5)$, $(10, 10)$-squares, and so on. We claim that they form an interlocking set of shadows which cover the entire y-axis. We have already seen that each such shadow has length $1\frac{3}{4}$. Furthermore, as we step from one square to the next along the diagonal, the shadow is raised by the same amount at each step. But we have already seen the lower endpoint of the shadow moved up from $y = \frac{1}{2}$ to $y = 1\frac{3}{4}$ in going from the $(5, 5)$-square to the $(10, 10)$-square, a shift of only $1\frac{1}{4}$. Hence all these shadows overlap, as claimed.

We are now able to conclude that no line with slope $\frac{3}{4}$ is able to thread the lattice. For any such line must intersect the y-axis at a point which is in the interior of some shadow. This shadow is cast by some square via a slope $\frac{3}{4}$ line; hence our line must have passed through the interior of this square, which means that it does not thread the lattice.

Part v: To begin, it is easy to confirm that the shadow cast by the $(0, 0)$-square stretches from $y = -\frac{p}{q}$ to $y = 1$. (Check this.) Now let the $(5c, 5d)$-square be an arbitrary square in the lattice. If this square is to cast a shadow which overlaps with that of the $(0, 0)$-square, then the line with slope $m = \frac{p}{q}$ and passing through $(5c + 1, 5d)$ (the lower right corner of the square) must have a y-intercept b satisfying $-\frac{p}{q} < b < 1$. We can determine the y-intercept by writing

$$y = mx + b \quad \Longrightarrow \quad 5d = \frac{p}{q}(5c + 1) + b \quad \Longrightarrow \quad b = 5d - \frac{p}{q}(5c + 1).$$

Hence we wish to find integers c and d such that

$$-\frac{p}{q} < 5d - \frac{p}{q}(5c + 1) < 1 \quad \Longrightarrow \quad 0 < dq - cp < \frac{p + q}{5}.$$

But p and q are relatively prime positive integers (we can safely assume that the fraction $\frac{p}{q}$ is written in lowest terms), hence there is a multiple of q which is exactly 1 greater than a multiple of p. (This is a standard fact from elementary number theory; you should test it out if you are unfamiliar with it.) Let dq and cp be these multiples, so that $dq - cp = 1$. Since $p + q > 5$, we have found a solution to the above inequality.

Part vi: Our argument will closely mimic part iv. Let c and d be the integers found in the previous part. The idea is that whenever we step from a square in the lattice to the square $5c$ units away in the x-direction and $5d$ units away in the y-direction, we reach a new square whose shadow is slightly higher than but overlaps with the previous shadow. The whole sequence of squares obtained in this manner will provide a set of interlocking shadows which completely cover the y-axis. Hence no line with slope $\frac{p}{q}$ can thread the lattice, using the same logic as above.

Finally, it turns out that whenever $p+q = 5$ there is a line with slope $m = \frac{p}{q}$ that threads the lattice, but only barely. And slopes for which $p + q < 5$ permit lines that cleanly thread the lattice, as the first part suggests. In case you were wondering, a line with an irrational slope can never thread the lattice. (Can you prove this?) Of course, there is a natural extension to lattices in which every k^{th} square is shaded, instead of every fifth square. The same sorts of results hold in this more general setting, a fact which the reader is invited to confirm.

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Round One Solutions

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