Team Play Topics

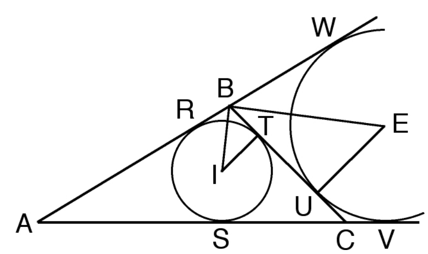
Round Three

A list of practice problems is provided below to aid in preparation for round three of the 2009 Mandelbrot Team Play. Note that these problems are not meant so serve as a precise indicator of the problems that will appear on the contest. However, students who understand how to solve them should be able to make significantly more progress than they might have otherwise. So work hard on the problems, and good luck on the Team Play!

Topics: Similar triangles, inscribed circles, tangency, right triangle trigonometry

Practice Problems

Given a triangle $\triangle ABC$, draw the incircle with center I tangent to its sides at points R, S and T as shown. Also draw the excircle with center E outside the triangle but within $\angle BAC$ that is tangent to the extended sides at points U, V and W. Let r be the radius of the incircle and let r_a be the radius of the excircle. Furthermore, let $s = \frac{1}{2}(a+b+c)$ be the semiperimeter of $\triangle ABC$ and define $s_a = s-a$, $s_b = s-b$ and $s_c = s-c$. (Here $a = BC$, $b = AC$, and $c = AB$ are the side lengths of $\triangle ABC$.)



1. Show that $s_b + s_c = a$ and that $s_a + s_b + s_c = s$.

2. It is a standard fact that the two tangent segments from a point to a circle are the same length. For example, we have $BW = BU$. Find as many such equalities as possible in this diagram.

3. We know that $AR + AS + BR + BT + CS + CT = a + b + c = 2s$. Use this fact to argue that $AS + SC + BT = s$ and deduce that $BT = s_b$. Finish by showing that $CT = s_c$.

4. Use the equal lengths from the second problem to show that $AV + AW = a + b + c$. Then deduce that $AV = s$ and hence that $CU = CV = s_b$. Finally, explain why $BU = s_c$.

5. Prove that $\triangle BIT \sim \triangle EBU$ and use this fact to demonstrate that $rr_a = s_b s_c$.

6. (Optional) Learn (or figure out) why the area K of $\triangle ABC$ is given by $K = rs$ and also by $K = r_a s_a$. Then use the previous problem to prove that $K = \sqrt{s s_a s_b s_c}$. (This famous area formula is known as Hero's formula.)

Hints and answers on the next page. $\Longrightarrow$

Round Three

1. We find $s_b + s_c = (s-b) + (s-c) = 2s - b - c = (a+b+c) - b - c = a$. Now adding s_a to both sides immediately gives $s_a + s_b + s_c = (s-a) + a = s$.

2. Examining points A, B and C in turn, we find $AR = AS$, $AV = AW$, $BR = BT$, $BU = BW$, $CS = CT$ and $CU = CV$.

3. Substituting $AR = AS$, $BR = BT$ and $CT = CS$ into the given equality we may deduce that $2(AS + SC + BT) = 2s$. Now divide by 2. But $AS + SC = b$; so subtracting b from both sides gives $BT = s - b = s_b$, as desired. Finally, we have $CT = BC - BT = a - s_b = s_c$. (See above.)

4. This is a very nice computation. We have

$$AV + AW = (AC + CV) + (AB + BW) = AC + AB + (CU + BU) = a + b + c.$$

But $AV = AW$ and $a + b + c = 2s$, so we deduce that $AV = AW = s$. Finally, subtracting $AC = b$ from both sides yields $CV = s - b$, so $CU = CV = s_b$. In the same way we have $BU = BW = AW - AB = s - c = s_c$.

5. It is well known that line BI is the angle bisector of $\angle ABC$ and that line BE is the angle bisector of $\angle WBU$. (Figure out why this is the case if you do not already know.) Letting $m\angle ABC = \beta$, we can write $m\angle IBT = \frac{1}{2}\beta$. Similarly,

$$m\angle EBU = \tfrac{1}{2}(m\angle WBU) = \tfrac{1}{2}(180^\circ - \beta) = 90^\circ - \tfrac{1}{2}\beta.$$

Therefore $m\angle BEU = \frac{1}{2}\beta$, since it is complementary to $\angle EBU$ due to the right angle at $\angle EUB$.

Thus triangles $\triangle EBU$ and $\triangle BIT$ both have one angle equal to $\frac{1}{2}\beta$ and one right angle, so they are similar. This gives the ratios

$$\frac{BT}{IT} = \frac{EU}{BU} \quad\Longrightarrow\quad (BU)(BT) = (EU)(IT) \quad\Longrightarrow\quad rr_a = s_b s_c.$$

6. Multiplying the equalities $K = rs$ and $K = r_a s_a$ together, we find that $K^2 = rr_a s s_a$. Then substituting $rr_a = s_b s_c$ yields $K^2 = s s_a s_b s_c$, or $K = \sqrt{s s_a s_b s_c}$, as desired.

Incidentally, these problems outline the nicest proof of Hero's formula of which I am aware. I first learned of this approach from John Conway.