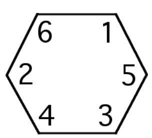
Answer Key

1. A	4. 1458
2. 58	5. -4.5
3. $m = 9,\ n = 7$	6. $(46, -13)$
	7. 66°

1. An equivalent way to phrase the question is to ask for which of the three graphs it is possible for a line to intersect the graph in three different points. Considered this way, it becomes obvious that this is possible for a sine curve but not for a parabola or natural log graph. Hence the answer is $\boldsymbol{A}$.

2. In order to obtain the smallest possible hexagon sum it makes sense to keep the numbers 4, 5, and 6 separated around the hexagon so that



these numbers are never multiplied together. Furthermore, one should surround the 6 by the 1 and 2 to avoid large multiples of the largest number. Testing the two possible cases, one discovers that placing the 1 next to the 5 gives a smaller hexagon sum, as shown. Hence the minimal value is $6 \cdot 1 + 1 \cdot 5 + 5 \cdot 3 + 3 \cdot 4 + 4 \cdot 2 + 2 \cdot 6 = \mathbf{58}$.

3. We rewrite the equation algebraically to obtain

$$\frac{m - 2n}{3m - 4n} = 5 \qquad \Longrightarrow \qquad m - 2n = 5(3m - 4n) = 15m - 20n.$$

Moving all terms involving m and n to separate sides of the equation then yields $18n = 14m$, or $9n = 7m$. It is now clear that we may take $\boldsymbol{m = 9}$ and $\boldsymbol{n = 7}$ as a solution. Any positive integer multiple of these numbers is also acceptable, such as $m = 18$ and $n = 14$, for instance.

4. There are six ways that Troy can place cars in the uppermost two spots; any of the three colors of car may park on the left side, leaving two remaining colors for the car on the right. Once a row is filled, there are exactly three ways to park two cars in the row beneath. For example, if the top row has cars with colors WB, then the next row could be BM, BW, or MW. So there are six ways to fill the first row, then three ways to fill the second row, then three ways to fill the third row, and so on, for a total of

$$6 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = \mathbf{1458}$$

parking arrangements. By the way, did you catch on to the fact that Troy sells vehicles produced by the Bavarian Motor Works?

5. We can add the two given fractions algebraically to obtain

$$\frac{A(x-1)}{(x+3)(x-1)} + \frac{6x}{(x+3)(x-1)} = \frac{6x + Ax - A}{(x+3)(x-1)}.$$

In order for the result to reduce to the form $B/(x-1)$, a factor of $(x+3)$ must cancel from the numerator. In other words, $6x + Ax - A$ must be a multiple of $(x+3)$. One way to check for this is to plug in $x = -3$ and require that the result vanish. Thus we need to have $-18 - 3A - A = 0$, which gives $A = \mathbf{-4.5}$. Sure enough, one can check that adding

$$\frac{-4.5}{x+3} + \frac{6x}{x^2 + 2x - 3} = \frac{1.5}{x-1}.$$

6. There is a delightfully short solution to this problem which does not require knowing which moves Sydney made to reach her final position. Let us say that to "add points" in the Cartesian plane means to add all the x-coordinates and y-coordinates separately. Thus the sum of the original points occupied by the markers is $(0,0) + (1,0) + (0,1) = (1,1)$. We now observe that no matter what moves Sydney makes, the sum of the three points on which the markers are located never changes. (This is called an *invariant* of the process.) Perhaps this is intuitively clear—we confirm this fact for the sake of completeness. If markers are located at (a_1, a_2) and (b_1, b_2), then after they jump over one another they occupy points $(2a_1 - b_1, 2a_2 - b_2)$ and $(2b_1 - a_1, 2b_2 - a_2)$. Sure enough, these two pairs of points have the same sum. Hence we need only find the point (x, y) for which $(63, -2) + (-108, 16) + (x, y) = (1, 1)$, which is quickly found to be $\mathbf{(46, -13)}$.

7. There are probably as many solutions to this problem as students who found the correct answer. Our approach illustrates the manner in which the problem was constructed. Using standard trigonometric identities, we first convert the expression to a sum of cosines for the sake of uniformity, yielding

$$\cos 30^\circ + \cos 42^\circ + \cos 102^\circ + \cos 174^\circ.$$

Note that 30°, 102°, and 174° are each separated by $72^\circ = \frac{1}{5}(360^\circ)$. We can continue this pattern by writing $\cos 42^\circ$ as $\cos(-42^\circ)$, which is then exactly 72° before 30°. Finally, we claim that

$$\cos(-42^\circ) + \cos 30^\circ + \cos 102^\circ + \cos 174^\circ + \cos 246^\circ = 0.$$

This follows from the fact that the sum of five equally spaced points (see previous solution) around the unit circle must be $(0, 0)$. (Proof?) The above identity is the special case of the x-coordinates of the five points located -42°, 30°, 102°, 174°, and 246° around the unit circle. Hence the original sum will equal $-\cos 246^\circ = \cos \mathbf{66}^\circ$. (Answers giving the same value, such as -66°, 294°, or 426° are also acceptable.)

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⋆ National Level ⋆

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